§ 2.7 Linear Elasto Statics

i, j, k, l are indicies associated with components. Repeated indicies sum 1 to n_{sd}

As with heat transfer we will make two passes through the weak form –
first: with indicial notation
second: with vectors and matrices

The transition between the two more complex because we
- will collapse 2nd order tensor's to vectors and a 4th order tensor to a matrix
- Change the definition of shear strain (just by a factor of 2)

Quantities of interest

u_i – displacement components – u is now a vector $\underset{\sim}{u}$
σ_{ij} – Cauchy stress tensor components
f_i – body force vector components
ϵ_{ij} – Green's strain tensor components

$$\epsilon_{ij} \equiv \underbrace{u_{(i,j)}}_{\text{symmetric derivative}} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

The stresses and strains are related through material parameters -
Generalized Hooke's Law (Constitutive eq.)

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

By definition $\epsilon_{kl} = \epsilon_{lk}$ symmetric

By balance of angular momentum (equilibrium of stress block)

$$\sigma_{ij} = \sigma_{ji} \quad \text{symmetric}$$

This leads to symmetrics in C_{ijkl}

$$C_{ijkl} = C_{klij}$$

$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{ijlk}$$

C_{ijkl} must also be pos. def. (at least for what we do)

$$C_{ijkl}\,\psi_{ij}\,\psi_{kl} \geq 0$$

$$C_{ijkl}\,\psi_{ij}\,\psi_{kl} = 0 \text{ iff } \psi_{ij} = \emptyset$$

all terms zero

Strong form

Given $f_i : \Omega \to \mathbb{R}$, $g_i : \Gamma_{g_i} \to \mathbb{R}$ and $h_i : \Gamma_{h_i} \to \mathbb{R}$,
find $u_i : \overline{\Omega} \to \mathbb{R}$ such that

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega \quad \text{(equilibrium)}$$

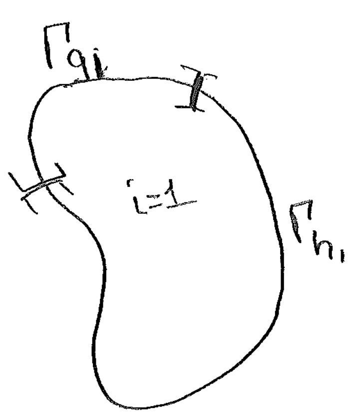
$$u_i = g_i \quad \text{on } \Gamma_{g_i} \qquad \text{displacement B.C.}$$

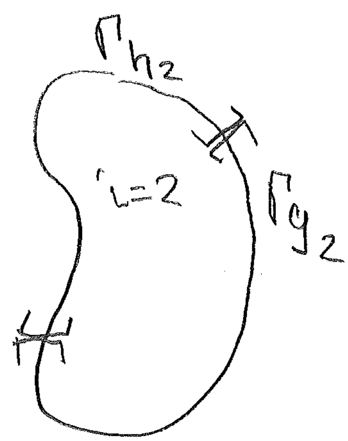
$$\sigma_{ij} n_j = h_i \qquad \text{traction B.C.}$$

where

$$\epsilon_{ij} \equiv u_{(i,j)} \equiv \frac{u_{i,j} + u_{j,i}}{2}$$

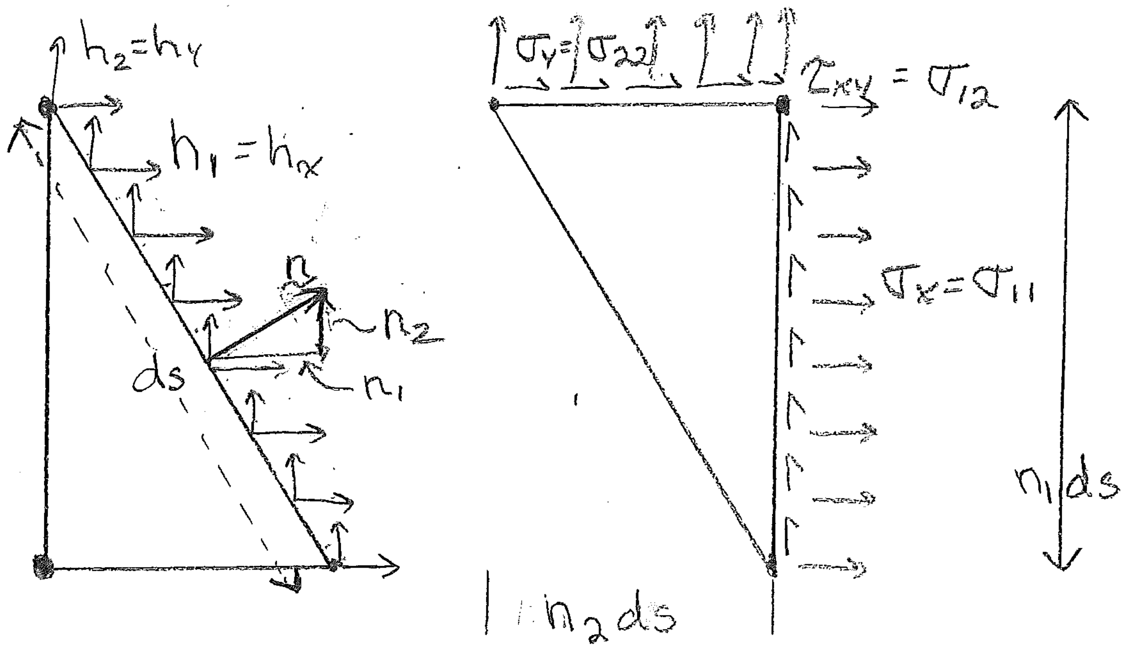
$$\sigma_{ij} = C_{ijkl} \, \epsilon_{kl}$$





$$\Gamma = \overline{\Gamma_{g_i} \cup \Gamma_{h_i}} \qquad i = 1(1)n_{sd}$$

$$0 = \Gamma_{g_i} \cap \Gamma_{h_i} \qquad i = 1(1)n_{sd}$$



To see espressions for h_1 and h_2 write statical equivalents

$\Sigma F_x = 0$

$$h_1 ds = \sigma_{11} n_1 ds + \sigma_{12} n_2 ds$$

$$h_1 = \sigma_{11} n_1 + \sigma_{12} n_2$$

$\Sigma F_y = 0$

$$h_2 ds = \sigma_{22} n_2 ds + \sigma_{12} n_1 ds$$

$$h_2 = \sigma_{22} n_2 + \sigma_{12} n_1$$

Applying MWR

$$\int_\Omega w_i(\sigma_{ij,j} - f_i)\, d\Omega = 0$$

integrate first term by parts

see Lemma 2 on pg 79 of text

Because of symmetry of σ_{ij}

$$-\int_\Omega w_{(i,j)}\, \sigma_{ij}\, d\Omega + \int_\Gamma w\sigma_{ij}\, d\Gamma - \int_\Omega w_i f_i\, d\Omega = 0$$

$$\int_\Gamma w\sigma_{ij}\, d\Gamma = \underbrace{\cancel{\int_{\Gamma_{g_i}} w_i\sigma_{ij}\, d\Gamma}}_{\to 0 \text{ for } w \in \mathcal{V}} + \int_{\Gamma_{h_i}} w_i \sigma_{ij}\, d\Gamma$$

$$= \int_{\Gamma_{h_1}} w_1 \sigma_{1j} n_j\, d\Gamma + \int_{\Gamma_{h_2}} w_2 \sigma_{2j} n_j\, d\Gamma \quad (n_{sd} = 2)$$

since $\Gamma_{h_1} \neq \Gamma_{h_2}$ in general - have two terms

another way to write it is

$$\sum_{\alpha=1}^{n_{sd}} \left(\int_{\Gamma_{h_\alpha}} w_\alpha \sigma_{\alpha j} n_j\, d\Gamma \right) = \sum_{\alpha=1}^{n_{sd}} \left(\int_{\Gamma_{h_\alpha}} w_\alpha h_\alpha\, d\Gamma \right)$$

no sum on α

Putting it to gether we can write the weak form -

Given $f_i : \Omega \to \mathbb{R}$, $g_i : \Gamma_{g_i} \to \mathbb{R}$ and $h_i : \Gamma_{h_i} \to \mathbb{R}$, find $u_i \in \delta_i$ such that

$$\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{\alpha=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_\alpha \underbrace{h_\alpha}_{\text{no sum in integral}} d\Gamma \right) \forall w_i \in \mathcal{V}_i$$

with $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$, $\epsilon_{ij} = u_{(i,j)} = \dfrac{u_{i,j} + u_{j,i}}{2}$

$$\delta_i = \{ u_i \mid u_i \in H^1,\ u_i = g_i \text{ on } \Gamma_{g_i} \} \quad i = 1(1) n_{sd}$$
$$\mathcal{V}_i = \{ w_i \mid w_i \in H^1,\ w_i = 0 \text{ on } \Gamma_{g_i} \} \quad i = 1(1) n_{sd}$$

Review text discussion of $(S) \to (W)$, $(W) \to (S)$

note- $\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_{(i,j)} C_{ijkl} \epsilon_{kl} \, d\Omega$

Stated with abstract form -

Given $\underset{\sim}{f}$, $\underset{\sim}{g}$ and $\underset{\sim}{h}$ (defined over right stuff) find $\underset{\sim}{u} \in \underset{\sim}{\delta}$ such that

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = (\underset{\sim}{w}, \underset{\sim}{f}) + (\underset{\sim}{w}, \underset{\sim}{h})_\Gamma \quad \forall\ \underset{\sim}{w} \in \underset{\sim}{\mathcal{V}}$$

where

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = \int_\Omega w_{(i,j)} C_{ijkl} \epsilon_{kl} \, d\Omega$$

$$(\underset{\sim}{w}, \underset{\sim}{f}) = \int_\Omega w_i f_i \, d\Omega$$

$$(\underset{\sim}{w}, \underset{\sim}{f})_\Gamma = \sum_{\alpha=1}^{n_{sd}} \int_{\Gamma_{h\alpha}} w_\alpha \underbrace{h_\alpha}_{\text{no sum.}} d\Gamma$$

We now work toward a "convenient" matrix description

we begin by representing the necessary terms from our second order tensors as terms listed in a vector –
(they still are second order tensors and transform coordinates like second order tensors – (not like vectors))

recalling that $\sigma_{ij} = \sigma_{ji}$

for $N_{sd} = 2$ $\quad \sigma_{12} = \sigma_{21}$, so only 3 independent terms $\leftarrow$ σ_{11}, σ_{22} and σ_{12} (or σ_{21})

Thus we will define

$$n_{sd} = 2 \quad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

for $n_{sd} = 3$ we have $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$
so $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}$ & σ_{23} can be the 6 independent terms

The text book ordering of the 6 terms

$$n_{sd} = 3 \quad \underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

Like wise we define a 3×1 strain vector for $n_{sd}=2$ and 6×1 strain vector for 3D
How we convert from Green's strains to "Engineering strains" (double "shear" terms).

$$n_{sd}=2 \qquad \underset{\sim}{\epsilon}(\underset{\sim}{u}) = \{\epsilon_I(\underset{\sim}{u})\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

$$n_{sd}=3 \qquad \underset{\sim}{\epsilon}(\underset{\sim}{u}) = \{\epsilon_I(\underset{\sim}{u})\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

We need our constitutive eq. in terms of the new stress and strain vectors

$$\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\epsilon}(u) \quad , \quad \underset{e\times 1}{\{\sigma\}} = \underset{e\times e}{[D]}\,\underset{e\times 1}{\{\epsilon(u)\}}$$

$[D]=[D]^T$ symmetric

$e = n_{sd}(n_{sd}+1)/2$, $e=3$ for $n_{sd}=2$, $e=6$ for $n_{sd}=3$

If we are given the terms C_{ijkl} using the known symmetries of C_{ijkl} and $\underset{\sim}{D}$ as well as the relations from σ_{ij} to $\underset{\sim}{\sigma}$ and ϵ_{ij} to $\underset{\sim}{\epsilon}(u)$ we can construct terms in $\underset{\sim}{D}$.
Text book shows how to do this
Since no one does this - I will not cover it or ask you to do it on home work or test.

with all this done we have

$$w_{(i,j)} c_{ijkl} u_{(k,l)} = \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})$$

$$a(\underset{\sim}{w},\underset{\sim}{u}) = \int_\Omega \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{D}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})\, d\Omega$$

Doing our normal decomposition of
$\underset{\sim}{u}^h = \underset{\sim}{v}^h + \underset{\sim}{g}^h$, $\underset{\sim}{u}^h \in \underset{\sim}{\delta}^h$, $\underset{\sim}{v}^h \in \underset{\sim}{\mathcal{V}}^h$, $\underset{\sim}{g}^h \in \delta^h$

where:

$\underset{\sim}{\mathcal{V}}^h = \{\underset{\sim}{w}^h \mid w_i^h \in \mathcal{V}_i^h\}$, $\mathcal{V}_i = \{w_i^h \mid w_i^h \in H^1, w_i^h = 0 \text{ on } \Gamma_{g_i}\}$

$\underset{\sim}{\delta}^h = \{\underset{\sim}{u}^h \mid u_i^h \in \delta_i^h\}$, $\delta_i = \{u_i^h \mid u_i^h \in H^1, u_i^h = g_i \text{ on } \Gamma_{g_i}\}$

we have our Galerkin form

Given $\underset{\sim}{f}$, $\underset{\sim}{g}$ and $\underset{\sim}{h}$ as before find $\underset{\sim}{u}^h = \underset{\sim}{v}^h + g^h \in \delta^h$ such that

$$a(\underset{\sim}{w}^h; \underset{\sim}{v}^h) = (\underset{\sim}{w}^h; \underset{\sim}{f}) + (\underset{\sim}{w}^h; \underset{\sim}{h})_\Gamma - a(w^h; g^h)$$

To get to the matrix form we have to deal with the fact that essential BC can be associated with any nodes (in general dof holders) ($n_{dof} \geq n_{sd}$ for vector systems)

Assuming nodal dof only where each node can have n_{dof},
lets use $n_{dof} = n_{sd}$ for the basic vector system

in this case the total possible number of dof is

$$n_{DOF} = n_{dof}\, \underbrace{n_{np}}_{\#\text{ of nodes}} = n_{sd}\, n_{np} \quad \text{for our current case}$$

n_{eq} = actual # dof in global system

$n_{eq} < n_{DOF}$ ⇐ must have sufficient essential B.C.

Consider again the set $\eta = \{1, 2, 3, \ldots, n_{np}\}$ an now define set

$$\eta_{g_i} = \{\text{nodes with a } g_i\}$$

with this we can write

$$v_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A\, d_{iA} \qquad i = 1(1)n_{sd}$$

$$g_i = \sum_{A \in \eta_{g_i}} N_A\, g_{iA} \qquad i = 1(1)n_{sd}$$

Introducing yet another construct: Euclidean basis vector $\underset{\sim}{e}_i$

$n_{sd} = 2 \qquad \underset{\sim}{e}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

$n_{sd} = 3 \qquad \underset{\sim}{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \underset{\sim}{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

with this we can write

$$\underset{\sim}{v}^h = v_i^h \underset{\sim}{e}_i \ , \quad \underset{\sim}{g}^h = g_i^h \underset{\sim}{e}_i \quad \left(\begin{array}{c}\text{remember to}\\ \text{sum}\end{array}\right)$$

$$\underset{\sim}{w}^h = w_i^h \underset{\sim}{e}_i \ , \quad w_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A c_{iA}$$

Note – notation still a bit messy but idea is to include the correct terms in each summation as well as accounting for components –

Substituting all this in and recalling the c_{iA}'s are arbitrary we have

$$\sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta - \eta_{g_i}} a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j)\, d_{jB} \right) =$$

(Using the dof $= n_{sd}$ case)

$$(N_A \underset{\sim}{e}_i, \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i, \underset{\sim}{h})_\Gamma$$

$$- \sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta_{g_i}} a(N_A \underset{\sim}{e}_i, N_B e_j)\, g_{jB} \right) \quad \begin{array}{l} A \in \eta - \eta_g \\ 1 \le i \le n_{sd} \end{array}$$

For writing the final matrix form we can use the ID device – now as a matrix

$$ID(i, A) \qquad \begin{array}{l} i = 1(1)\, n_{sd} \\ A = 1(1)\, n_{np} \end{array} \quad \left(\begin{array}{c}\text{vector system}\\ \text{nodal dof}\\ \text{only}\end{array}\right)$$

```
P = 0
Do A = 1 to n_np
   Do i = 1 to n_sd
      If A ∈ η - η_gi
         P = P + 1
         ID(i, A) = P      ← equation # in K
      else
         ID(i, A) = 0
      end if
   end Do
end Do
n_eq = P   ⇐ total number of actual equations
```

Using this we have

$$[K]\{d\} = \{F\} \qquad \left(\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F} \right)$$

$$P = ID(i,A) \quad , \quad Q = ID(j,B)$$

$$K_{PQ} = a(N_A \underset{\sim}{e}_i \, , \, N_B \underset{\sim}{e}_j)$$

$$F_P = (N_A \underset{\sim}{e}_i \, , \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i \, \underset{\sim}{h})_{\Gamma} - \sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta_{g_j}} a(N_A \underset{\sim}{e}_i \, , N_B \underset{\sim}{e}_j) \, g_{jB} \right)$$

⇒ We will see exactly how terms go into place when we look at our pseudo code later

To complete the ability to use vector matrix form we want to introduce the "B" matrix again

$$\underset{\sim}{\epsilon}(N_A \underset{\sim}{e_i}) = \underset{\sim}{B_A}\, \underset{\sim}{e_i} \Leftarrow \text{Strain-displacement operator}$$

for $n_{sd}=2$ $\quad \underset{\sim}{B_A} = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix}$

3×2 ← n_{sd}; 3 = # strain components

for $n_{sd}=3$ $\quad \begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix}$

6×3

With this we can write (not very useful!)

$$K_{PQ} = \underset{\sim}{e_i}^T \int_\Omega \underset{\sim}{B_A}^T \underset{\sim}{D}\, \underset{\sim}{B_B}\, d\Omega\, \underset{\sim}{e_j}$$

and noting that $\int_\Omega N_A \underset{\sim}{e_i}\, \underset{\sim}{f}\, d\Omega = \int_\Omega N_A f_i\, d\Omega$

$$\int_\Gamma N_A \underset{\sim}{e_i}\, \underset{\sim}{h}\, d\Gamma = \int_{\Gamma_{h_i}} N_A h_i\, d\Gamma$$

$$F_P = \int_\Omega N_A f_i \, d\Gamma + \int_{\Gamma_{h_i}} N_A h_i \, d\Gamma - \underbrace{\sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta_{g_j}} a(N_A \underset{\sim}{e_i}, N_B e_j)\, g_{jB} \right)}_{\text{still messy} -}$$

Is $K_{PQ} = K_{QP}$ (symmetric)

$$K_{PQ} = \left(\underset{\sim}{e_i}^T \int_\Omega \underset{\sim}{B_A}^T \underset{\sim}{D} B_B \, d\Omega \, e_j \right)^T \leftarrow \text{since its just a "number"}$$

$$= e_j^T \int_\Omega \underset{\sim}{B_B}^T \underset{\sim}{D}^T \underset{\sim}{B_A} e_i \qquad \text{with } D = D^T$$

$$= e_j^T \int_\Omega \underset{\sim}{B_B}^T D \, B_A e_i \equiv K_{QP}$$

Positive-Definiteness of K

$$\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} \geq 0, \quad \underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0 \text{ iff } \underset{\sim}{c} = \phi$$

$$\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = \sum_{P,Q=1}^{n_{eq}} c_P K_{PQ} c_Q$$

$$= \sum_{i,j=1}^{n_{dof}} \left(\sum_{\substack{A \in \eta - \eta_{g_i} \\ B \in \eta - \eta_{g_j}}} c_{iA} \, a(N_A \underset{\sim}{e_i}, N_B \underset{\sim}{e_j}) \, c_{jB} \right)$$

use bilinearity

$$= a\left(\sum_{i=1}^{n_{dof}} \left(\sum_{A \in \eta - \eta_{g_i}} c_{iA} N_A \underset{\sim}{e_i} \right), \sum_{j=1}^{n_{dof}} \left(\sum_{B \in \eta - \eta_{g_j}} c_{jB} N_B \underset{\sim}{e_j} \right) \right)$$

$$= a(w^h, w^h) = \int_\Omega \underbrace{w^h_{(i,j)} C_{ijkl} w^h_{(k,l)}}_{\geq 0 \text{ for posdef } C_{ijkl}} \, d\Omega$$

we have positiveness

assume $\underset{\sim}{c}^T \underset{\sim}{K} \underset{\sim}{c} = 0$

That requires $w^h_{(i,j)} C_{ijkl} w^h_{(kl)} = 0$

This requires $w^h_{(i,j)} = 0$ <u>for pos. def. C_{ijkl}</u>

There are a limited set of $\underset{\sim}{w}^h \neq 0$ for which $w^h_{(i,j)} = \dfrac{w^h_{i,j} + w^h_{j,i}}{2} = 0$

The correspond

for $n_{sd}=2$ $\underset{\sim}{w}^h(x) = \underset{\sim}{c} + c_3 \underbrace{(x_1 \underset{\sim}{e}_2 - x_2 \underset{\sim}{e}_1)}$

rigid body rotation in plane

rigid body translation

for $n_{sd}=3$ $\underset{\sim}{w}^h(x) = \underset{\sim}{c}_1 + \underset{\sim}{c}_2 \times \underset{\sim}{x}$ ← Rigid body rotations

cross product

rigid body translation

$\underset{\sim}{K}$ is pos.-def. so long as rigid body motions are constrained by the prescribed $\underset{\sim}{g}^h$

Finally – We do not want to do those crazy integrals to directly get K_{PQ} and F_P terms

We want to do elements – the whole element – at once and have the assembly operator take care of putting things in the right place the right way

$$\underset{\sim}{K} = \overset{n_{el}}{\underset{e=1}{A}} \underset{\sim}{k}^e \;, \quad \underset{\sim}{F} = \overset{n_{el}}{\underset{e=1}{A}} (\underset{\sim}{k}^e ; \underset{\sim}{f}^e)$$

n_{en} = # nodes / element

n_{ed} = # dof / node = n_{sd} for simple vector systems

$n_{ee} = n_{en} n_{ed}$ = # of dof of the element
→ will be the size of element $\underset{\sim}{k}^e, \underset{\sim}{f}^e$

$$\underset{\sim}{k}^e = [k^e_{pq}] \;, \quad \underset{\sim}{f}^e = \{f^e\} \qquad 1 \le p, q \le n_{ee}$$

$$k_{pq} \; \underset{\sim}{e}_i^T \int_{\Omega_e} \underset{\sim}{B}_a^T \underset{\sim}{D} \, \underset{\sim}{B}_b \, d\Omega \, \underset{\sim}{e}_j \qquad \begin{aligned} p &= n_{ed}(a-1) + i \\ q &= n_{ed}(b-1) + j \end{aligned}$$

for $n_{sd} = 2$

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}$$

for $n_{sd} = 3$

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix}$$

$$f_p^e = \underbrace{\int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma_{h_i}^e} N_a h_i \, d\Gamma}_{\text{what I called } \hat{f}_p^e} - \sum_{q=1}^{n_{ee}} k_{pq} g_q^e$$

$$\Gamma_{h_i}^e = \Gamma_{h_i} \cap \Gamma^e$$

portion of elements

$$g_q = g_j(\underset{\sim}{x}_b^e) \text{ if node is on } \Gamma_{g_j}$$

is zero otherwise

In actual programs we avoid the $\underset{\sim}{e}_i$'s and just do the whole thing at once for example –

Define $\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2 \cdots \underset{\sim}{B}_{n_{en}}]$

then
$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$$

Stress evaluation

$$\underset{\sim}{\epsilon}(\underset{\sim}{u}^h(\underset{\sim}{x})) = \sum_{a=1}^{n_{en}} \underset{\sim}{B}_a \underset{\sim}{d}_a^e$$

$\underset{\sim}{d}_a^e$ ← dof at a node

$$\underset{\sim}{\sigma}(\underset{\sim}{x}) = \underset{\sim}{D}(\underset{\sim}{x}) \underset{\sim}{\epsilon}(\underset{\sim}{u}^h(\underset{\sim}{x})) = \underset{\sim}{D}(\underset{\sim}{x}) \sum_{a=1}^{n_{en}} \underset{\sim}{B}_a \underset{\sim}{d}_a^e$$

$$= \underset{\sim}{D}(\underset{\sim}{x}) \underset{\sim}{B} \underset{\sim}{d}^e$$

where

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \underset{\sim}{B}_3, \cdots \underset{\sim}{B}_{n_{en}}], \quad \underset{\sim}{d} = \begin{Bmatrix} \underset{\sim}{d}_1^e \\ \underset{\sim}{d}_2^e \\ \vdots \\ \underset{\sim}{d}_{n_{en}}^e \end{Bmatrix}$$

Before seeing how this all fits together in the form of pseudo code to a very basic F.E. code, we need one more tool – Numerical Integration (will cover Chapt. 3)

$$\underset{\sim}{k}^e = \int_\Omega \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$$

is often messy because of mappings –

Thus we will apply numerical integration

$$\underset{\sim}{k}^e = \sum_{l=1}^{n_{int}} W_l \, \underset{\sim}{B}^T(\underset{\sim}{x}_l) \, \underset{\sim}{D}(\underset{\sim}{x}_l) \, \underset{\sim}{B}(\underset{\sim}{x}_l) + R^{int}$$

Since we do not know R^{int} past being able to say it small enough when we use a sufficient number of properly selected integration points, we will drop it and use

$$\underset{\sim}{k}^e \simeq \sum_{l=1}^{n_{int}} W_l \, \underset{\sim}{B}^T(\underset{\sim}{x}_l) \, \underset{\sim}{D}(\underset{\sim}{x}_l) \, \underset{\sim}{B}(\underset{\sim}{x}_l)$$