Hughes Chapter II 2D & 3D

Consider basic heat transfer and elasticity

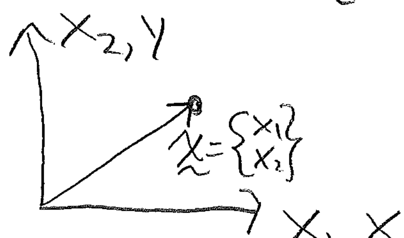
We will cover § 2.1–2.5 and 2.7–2.9
The material from 2.6 and 2.10 will be covered in a set of notes to be covered at the end of the chapter

Notation

n_{SD} – # of spatial dimensions (2 or 3)
$\Omega \subset \mathbb{R}^{n_{sd}}$ – domain
Γ – boundary , $\bar{\Omega} = \Omega \cup \Gamma$
$\underset{\sim}{x}$ – location in Ω
$\underset{\sim}{n}$ – normal vector

$n_{sd}=2$ $\underset{\sim}{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$, $\underset{\sim}{n} = \{n_i\} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$

$n_{sd}=3$ $\underset{\sim}{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$, $n = \{n_i\} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$



$\Gamma_g = 0,\ \Gamma_{g_i} \neq 0$

u – scalar field
$\underset{\sim}{u}$ – vector field

$\Gamma = \bar{\Gamma}_g \cup \bar{\Gamma}_h$ for scalar field , $\Gamma = \bar{\Gamma}_{g_i} \cup \bar{\Gamma}_{h_i}$ for vector field

$0 = \Gamma_g \cap \Gamma_h$ for scalar field , $0 = \Gamma_{g_i} \cap \Gamma_{h_i}$ for vector field

$$u_{,i} = u_{,x_i} = \frac{\partial u}{\partial x_i}, \quad u_{,ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

$$u_{,ii} = u_{,11} + u_{,22} + u_{,33} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

remember summation convention –
repeated subscripts sum

Divergence –

$$\int_\Omega f_{,i} \, d\Omega = \int_\Gamma f n_i \, d\Omega$$

note – things can get complicated

$$\int_\Omega (t_{ij} S_i)_{,j} \, d\Omega = \int_\Omega \underbrace{t_{ij,j} S_i + t_{ij} S_{i,j}}_{\text{by product}} \, d\Omega =$$

$$= \underbrace{\int_\Gamma (t_{ij} S_i) n_j \, d\Omega}_{\text{by divergence}}$$

note – in all cases the i's and j's sum out

Integration by parts

$$\int_\Omega f_{,i} \, g \, d\Omega = -\int_\Omega f g_{,i} \, d\Omega + \int_\Gamma f g n_i \, d\Gamma$$

$$\int_\Omega (fg)_{,i} \, d\Omega = \int_\Omega (f g_{,i} + f_{,i} g) \, d\Omega \xrightarrow[\text{divergence}]{\text{used}} \int_\Omega (fg)_{,i} \, d\Omega$$

§ 2.3 Linear, Steady State, Heat Conduction

f - heat generated per unit volume

q_i - components of heat flux

u - temperature

Fourier Law

$k_{ij} u_{,j} + q_i = 0$ in Ω ⇐ constitutive equation

$k_{ij} = k_{ji}$ ⇐ symmetric heat conduction tensor

if an isotropic material - $k_{ij} = k\,\delta_{ij}$ (diag.)

Heat Conduction

$q_{i,i} = f \Rightarrow q_{1,1} + q_{2,2} + q_{3,3} = f$

Strong form

Given $f: \Omega \to \mathbb{R}$, $g: \Gamma_g \to \mathbb{R}$ and $h: \Gamma_h \to \mathbb{R}$, find $u: \bar{\Omega} \to \mathbb{R}$ such that

$q_{i,i} = f$ in Ω

$u = g$ on Γ_g

$-q_i n_i = h$ on Γ_h

note - we will put $q_i = -k_{ij} u_{,j}$ into heat eq. later
will give 2nd derivative on u

Apply MWR

$$\int_\Omega w\,(q_{i,i} - f)\,d\Omega = 0 \quad \forall w \in \mathcal{V}$$

Integrate first term by parts, we get

$$-\int_\Omega w_{,i}\, q_i \,d\Omega + \int_\Gamma w\, q_i n_i \,d\Gamma - \int w f \,d\Omega = 0 \;\forall w \in \mathcal{V}$$

$$-\int_\Omega w_{,i}\, q_i \,d\Omega + \underbrace{\cancel{\int_{\Gamma_g} w\, q_i n_i \,d\Gamma}}_{0 \text{ if } u=g \text{ a priori}} + \int_{\Gamma_h} w(-h)\,d\Gamma - \int w f\,d\Omega = 0 \quad \forall w \in \mathcal{V}$$

Gives us the weak form
(dropping all the $\to \mathbb{R}$'s)

Given $f:\Omega$, $g:\Gamma_g$, and $h:\Gamma_n$, find $u \in \mathcal{S}$ such that

$$-\int_\Omega w_{,i}\, q_i \,d\Omega = \int_\Omega w f \,d\Omega + \int_{\Gamma_h} w h \,d\Gamma \quad \forall w \in \mathcal{V}$$

$$\mathcal{V} = \{ w \mid w \in H^1,\; w|_{\Gamma_g} = 0 \}$$

$$\mathcal{S} = \{ u \mid u \in H^1,\; u|_{\Gamma_g} = g \}$$

and $q_i = -k_{ij}\, u_{,j}$ will be used.

Define

$$a(w,u) = -\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w_{,i} K_{ij} u_{,j} \, d\Omega$$

$$(w,f) = \int_\Omega w f \, d\Omega$$

$$(w,h)_\Gamma = \int_{\Gamma_h} w h \, d\Gamma$$

then we get the standard weak form equation -

$$a(w,u) = (w,f) + (w,h)_\Gamma$$

The indicial notation is the best form for the raw equation construction, manipulation and understanding - but not most obvious for basic transformation to computer code that likes to think of scalars, vectors and matrices only

Therefore we will write all of this with a change of notation.

$$\underset{\sim}{\nabla} u = \{u_{,i}\} = \begin{Bmatrix} u_{,1} \\ u_{,2} \\ u_{,3} \end{Bmatrix}, \quad \underset{\sim}{\nabla} w = \{w_{,i}\} = \begin{Bmatrix} w_{,1} \\ w_{,2} \\ w_{,3} \end{Bmatrix}$$

$$\underset{\sim}{K} = [K_{ij}] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}_{3\times3}$$

$n_{sd} = 2 \leftarrow 2\times2$

for isotropic and homogeneous $\underset{\sim}{K} = K \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

With this you can confirm

$$w_{,i} K_{ij} u_{,j} = (\underset{\sim}{\nabla} w)^T \underset{\sim}{K} (\underset{\sim}{\nabla} u)$$

$$a(w,u) = \int_\Omega (\underset{\sim}{\nabla} w)^T \underset{\sim}{K} (\underset{\sim}{\nabla} u)\, d\Omega$$

Galerkin form - straight forward
$u^h = v^h + g^h$ so we get

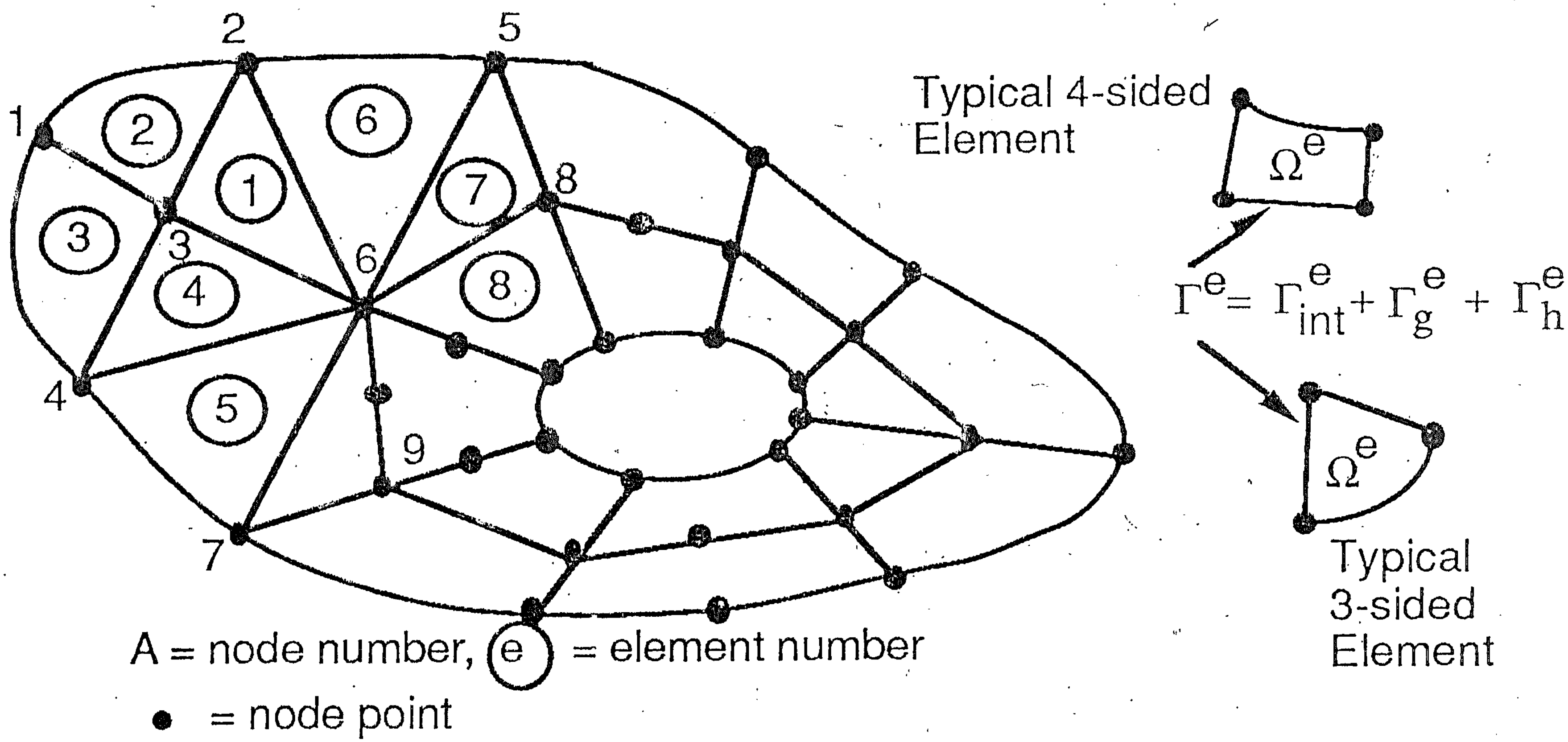
Given f, g, h as before, find $u^h = v^h + g^h \in \delta^h$
such that

$$a(w, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h)$$

For now lets view things at the global level and expand

$$w^h(\underset{\sim}{x}) = \sum_{A \in \eta - \eta_g)} N_A(\underset{\sim}{x}) c_A$$

η - set of all nodes (assuming nodal dof only)
η_g - set of nodes with essential BC applied
$\eta = \{1, 2, 3, \dots n_{np}\}$ n_{np} = # of node points
$\eta_g \subset \eta$



$$\bar{\Omega} = \bigcup_e \bar{\Omega}^e$$

union over all elements

Some other terms we will use

n_{dof} = # of dof (unknowns) per node

For our current case of a scalar unknown and interpolating shape functions
$n_{dof} = 1$

In the case of a vector field and interpolating shape functions
$n_{dof} = n_{sd}$

There are other options with $n_{dof} \geq n_{sd}$ used for some vector field cases.

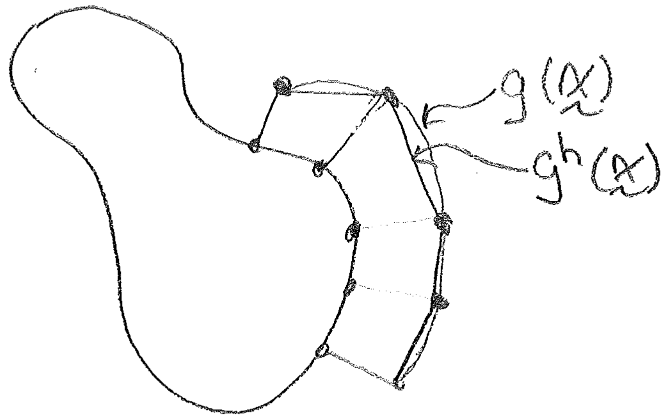
n_{DOF} = total number of possible dof (case of no essential BC – or if you will – before accounting for them) (Text book does not have this term.)

(back to Galerkin form)

$$v^h(\underset{\sim}{x}) = \sum_{A \in (\eta - \eta_g)} N_A(\underset{\sim}{x})\, d_A$$

$$g^h(\underset{\sim}{x}) = \sum_{A \in \eta_g} N_A(\underset{\sim}{x})\, g_A$$

$g_A = g(\underset{\sim}{x}_A)$ assuming interpolating shape functions



Substituting into Galerkin form we get

$$\sum_{B \in \eta - \eta_g} a(N_A, N_B)\, d_B = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)\, g_B$$

$$A \in \eta - \eta_g$$

Since we are not requiring the set η_g to be first or last set of nodes in list – we need a mechanism – The Designation matrix $ID_{n_{dof} \times n_{np}}$

for current case $n_{dof} = 1$ so ID is just a vector

$\Rightarrow$ Construct ID

```
P=0
Do A = 1 to n_np
   If A ∈ η - η_g then
        P = P+1        ← Label equation # in [K]
        ID(A)=P
   else
        ID(A)=0
   end if
end Do
n_eq=P
```

we will see an extended ID later –
(extended to do better with essential BC contributions to RHS)

This produces

$$[K]_{n_{eq}\times n_{eq}} \{d\}_{n_{eq}} = \{F\}_{n_{eq}}$$

$$K_{PQ} = a(N_A, N_B) \quad \text{for } P = ID(A) \neq 0,\ 1 \leq P \leq n_{eq}$$
$$Q = ID(B) \neq 0,\ 1 \leq Q \leq n_{eq}$$

$$F_P = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)\, g_B$$

[K] is symmetric

$$K_{PQ} = a(N_A, N_B) = a(N_B, N_A) = K_{QP}$$

[K] is positive definate so long as
[K] is positive def. and $\Gamma_g \neq 0$

proof follows same process as we saw for 1D case

Element stiffness matrix

It is not convenient to directly calculate the terms of the global stiffness matrix, K_{PQ}, and force vector, F_P.

Instead it is more convienient to constrct element stiffness matrices and load vectors and to "assemble" their contributions into the global system.

The method given in the text book does not provide an entirely clean method for accounting for non-zero essential B.C. Thus I will skip sections 2.6 - Data processing for heat conduction and 2.10 - Data processing for elastostatics. Instead we will cover some notes that slightly extend the method presented in those sections.

For now lets leave it a bit more abstract

$$\underset{n_{eq}\times n_{eq}}{[K]} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \underset{n_{ee}\times n_{ee}}{[k^e]}$$

$$\underset{n_{eq}\times 1}{\{F\}} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} \left(\underset{n_{eq}\times 1}{\{f^e\}}, [k^e]\right)$$

$\mathbf{A}$ → assembly operator to be given later

n_{eq} = # of equations (dof) in complete system

n_{ee} = # of element possible dof

$\{f^e\}$ - accounts for body forces & tractions

$[k^e]$ - needed for non-zero essential BC

At the element level the text discusses the element contributions

$$[k^e] = \underset{\sim}{k}^e = \left[k^e_{ab}\right]_{n_{ee}\times n_{ee}}, \quad \underset{\sim}{f}^e = \{f^e_a\}_{n_{ee}\times 1} \quad 1 \le a,b \le n_{ee}$$

for the scalar case and interpolating shape functions $n_{ee} = n_{en} \Rightarrow$ #nodes/element

$$k^e_{ab} = a(N_a, N_b) = \int_{\Omega^e} (\underset{\sim}{\nabla} N_a)^T \underset{\sim}{\kappa} (\underset{\sim}{\nabla} N_b)\, d\Omega$$

The text book writes

$$f^e_a = \underbrace{\int_{\Omega^e} N_a f\, d\Omega + \int_{\Gamma^e_h} N_a h\, d\Gamma}_{\text{This is what I called } \{\hat{f}^e\} \text{ on previous page}} - \underbrace{\sum_{b=1}^{n_{ee}} k^e_{ab} g_b}_{\text{The Assembly operation will take care of this part}}$$

A combined expression for k^e in terms of matrices

$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}\, d\Omega \quad \text{(we like to get them to this form)}$$

Where

$$\underset{\sim}{D}_{n_{sd}\times n_{sd}} = \underset{\sim}{k} = [K]$$

The construction of $\underset{\sim}{B}$ is done such that the multiplication works out correctly considering the current interpolating shape function case

$n_{ee} = n_{en}$ = # of nodes/element

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \underset{\sim}{B}_3, \ldots \underset{\sim}{B}_{n_{en}}]_{n_{sd}\times n_{en}}$$

$$\underset{\sim}{B}_{a\,n_{sd}\times 1} = \underset{\sim}{\nabla} N_a$$

you can confirm single terms -

$$k^e_{ab} = \int_{\Omega^e} \underset{\sim}{B}_a^T \underset{\sim}{D} \underset{\sim}{B}_b \, d\Omega$$

Calculation of heat fluxes - $q_i = -k_{ij} u_{,j}$

$$u^e = \sum_{a=1}^{n_{ee}} N_a d^e_a$$

$\underset{\sim}{d}^e$ - contains all possible dof's in general dof and g's

$$u^e_{,j} = \sum_{a=1}^{n_{ee}} N_{a,j} d^e_a$$

thus

$$q^e_i = -k_{ij} \sum_{a=1}^{n_{ee}} N_{a,j} d^e_a$$