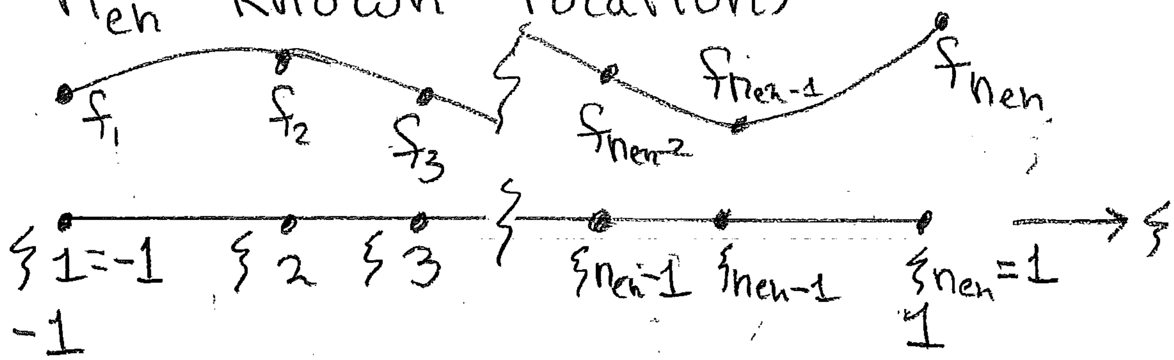
# Lagrange Interpolation

A procedure to construct an order $n_{en}-1$ polynomial that interpolates a 1D function at $n_{en}$ points

Starting point: knowledge of the value of the function at $n_{en}$ known locations



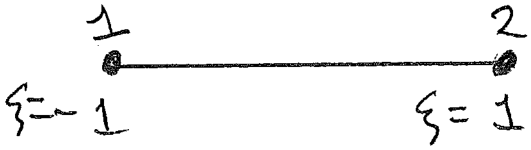

Lagrange Polynomials: An order $n_{en}-1$ order polynomial that interpolates at the $n_{en}$ given locations

Order of

$$f(\xi) = \sum_{a=1}^{n_{en}} l_a^{n_{en}-1}(\xi)\, f(\xi_a) \quad , \text{note } l_a^{n_{en}-1}(\xi_b) = \delta_{ab}$$

$$l_a^{n_{en}-1}(\xi) = \frac{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi-\xi_b)}{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi_a-\xi_b)}$$

## Two node linear




$n_{en}=2$

$n_{en}-1=1 \Leftarrow$ linear functions

$$l_1^1(\xi)=\frac{\prod_{\substack{b=1\\b\neq 1}}^{2}(\xi-\xi_b)}{\prod_{\substack{b=1\\b\neq 1}}^{2}(\xi_1-\xi_b)}=\frac{\xi-\xi_2}{\xi_1-\xi_2}=\frac{\xi-1}{-1-1}=\frac{1}{2}(\xi-1)$$

$$l_2^1(\xi)=\frac{\prod_{\substack{b=1\\b\neq 2}}^{2}(\xi-\xi_b)}{\prod_{\substack{b=1\\b\neq 2}}^{2}(\xi_2-\xi_b)}=\frac{\xi-\xi_1}{\xi_2-\xi_1}=\frac{\xi-(-1)}{1-(-1)}=\tfrac{1}{2}(\xi+1)$$

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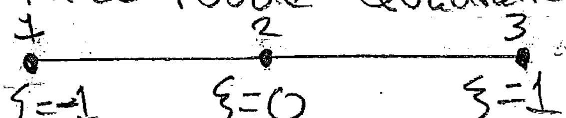
## Three Node Quadratic (equially spaced)




$n_{en}=3$ , $n_{en}-1=2$

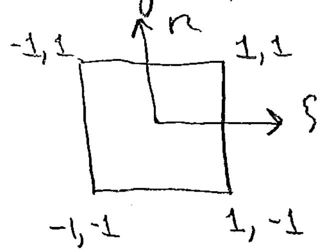
$$l_1^2(\xi)=\frac{\prod_{\substack{b=1\\b\neq 1}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\b\neq 1}}^{3}(\xi_1-\xi_b)}=\frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}=\frac{\xi(\xi-1)}{(-1)(-2)}=\frac{1}{2}\xi(\xi-1)$$

$$l_2^2(\xi)=\frac{\prod_{\substack{b=1\\b\neq 2}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\b\neq 2}}^{3}(\xi_2-\xi_b)}=\frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}=\frac{(\xi-(-1))(\xi-1)}{(+1)(-1)}=1-\xi^2$$

$$l_3^2(\xi)\ \frac{\prod_{\substack{b=1\\b\neq 3}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\b\neq 3}}^{3}(\xi_3-\xi_b)}=\frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}=\frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)}=\frac{1}{2}\xi(\xi+1)$$

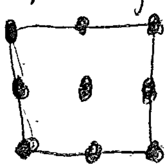
# Constructions of 2-D - 4 sided elements Using Lagrange Shape functions.

As indicated previously, the element's shape functions will be written in the parametric coordinate system - Therefore, all elements are bi-unit squares



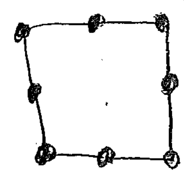

We will consider two families of such elements

(Full) Lagrangian



Will in general have interior and boundary nodes

Serendipity



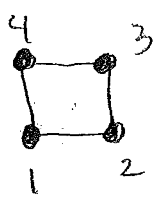
- Nodes only on edges

Also → Variable # nodes

Start with Lagrangian -

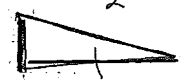
Elements defined by multiplying 1-D Lagrange polynomials in one direction times those in the other direction(s) 3-D

Start with simple 4-noded element



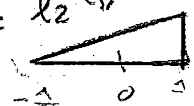

With 2 nodes per side we want to use linear

Consider the 1-D Lagrange Polynomials needed - Note - 2 nodes per edge - Linear in $\xi$ direction - $l_1^1 = \frac{1}{2}(1-\xi)$

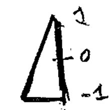



$= l_1^1(\xi)$

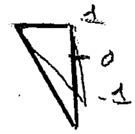
$l_2^1 = \frac{1}{2}(1+\xi) = l_2^1(\xi)$



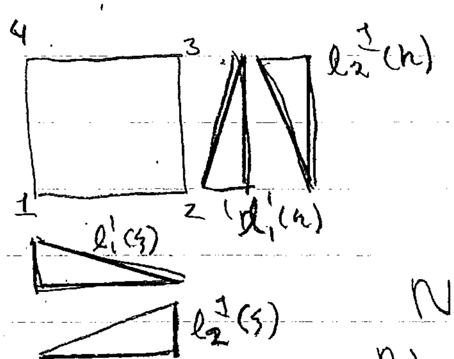

in $\eta$ direction - $l_1^1 = \frac{1}{2}(1-\eta) = l_1^1(\eta)$




$l_2^2 = \frac{1}{2}(1+\eta) = l_2^2(\eta)$




the shape functions at the nodes are the "correct multiplication of these




$$u^h = \sum_{a=1}^{4} N_a d_a$$

$$N_1 = l_1^1(\xi)\, l_1^1(\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = l_2^1(\xi)\, l_1^1(\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = l_2^1(\xi)\, l_2^1(\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = l_1^1(\xi)\, l_2^1(\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Note - $N_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta) = \frac{1}{4}(1+\xi_a\xi+\eta_a\eta+\xi_a\eta_a\xi\eta)$

in other words each $a_{11} = -1$

$$N_a = a_{0a} + a_{1a}\xi + a_{2a}\eta + a_{3a}\xi\eta$$

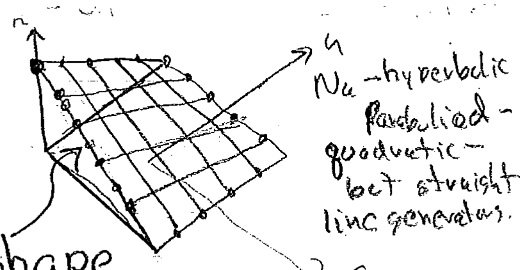
Since each shape function has same form

$$u^h = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$$

Pascals triangle (in $\xi$-$\eta$)

this $\xi$ | this times $\eta$

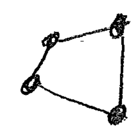
$\xi^2$ $\xi\eta$ $\eta^2$



– How does this element do on meeting conditions C1 - C3 ⇐ assume $m=1$

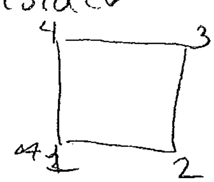
C1 – Intra element continuity (Smoothness on $\Omega^e$)

If $\underset{\sim}{x}$ happens to be same as $\xi, \eta$
Then fine can take all first partial derivatives
$\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}$

– Now if in $\underset{\sim}{x}$ element looks like 
Then we must understanding the mapping between systems better – Bottom line, for straight-sided need corner angles $> 0°$ and $< 180°$ – more later

C2 – Inter element continuity – need $C^0$ continuous across $\Gamma^e$

– Lets consider the values of the shape functions on a element edge –
Consider edge 1-2




$$\eta = -1$$

$$N_1\big|_{1-2} = \tfrac{1}{4}(1-\xi)(1-(-1)) = \tfrac{1}{2}(1-\xi)$$

$$N_2\big|_{12} = \tfrac{1}{2}(1+\xi)$$

$$N_3\big|_{1-2} = \tfrac{1}{4}(1+\xi)(1+(-1)) = 0$$

$$N_4\big|_{12} = 0$$

$$u^h\big|_{1-2} = \sum_{a=1}^{4} N_a\big|_{1-2} d_a = \tfrac{1}{2}(1-\xi)\, d_1 + \tfrac{1}{2}(1+\xi)\, d_2 + 0\, d_3 + 0\, d_4$$

Note – it is linear function of only the nodal values at the two nodes on that edge

Since neighboring elements share the same two nodal values – it will be $C^0$ continuous between elements – no gaps can open

Same for all edges

C3 Completeness

Consider in $\xi$-$\eta$

from before $u^h = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$

to meet completeness must have at least through linear $u^h = a_0 + a_1\xi + a_2\eta$ ⟸ have it – OK

What about in $\underline{x}$

as before require

$$u^h = \sum_{a=1}^{n_{en}} N_a d_a^e = c_0 + c_1 x + c_2 y$$

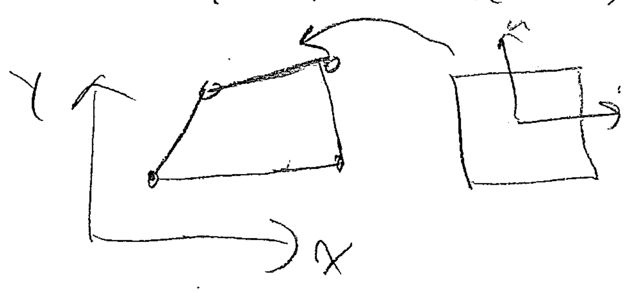
as before

$$u^h\Big|_{\substack{x=x_a \\ y=y_a}} = d_a^e = c_0 + c_1 x_a + c_2 y_a$$ ⟸ Put back into above eq.

$$u^h = \sum_{a=1}^{n_{en}} N_a (c_0 + c_1 x_a + c_2 y_a) = c_0 \sum N_a + c_1 \sum N_a x_a + c_2 \sum N_a y_a$$

set if
$$\sum N_a = 1$$
$$\sum N_a x_a = x$$
$$\sum N_a y_a = y$$

$$\sum N_a = \tfrac{1}{4}(1-\xi)(1-\eta) + \tfrac{1}{4}(1+\xi)(1-\eta) + \tfrac{1}{4}(1+\xi)(1+\eta) + \tfrac{1}{4}(1-\xi)(1+\eta)$$
$$= 1$$




Note defining
$$x = \sum N_a x_a$$
$$y = \sum N_a y_a$$
is a convient way to map between $\xi$ $\eta$ and $\underline{x}$

We will talk a bit more about such maps later –

# Higher Order Lagrange elements

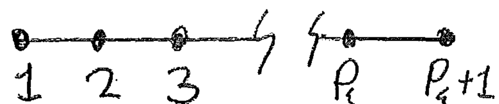
Products of 1-D shape functions

2D $\quad N_a(\xi,\eta) = l_{a_\xi}^{p_\xi}(\xi)\; l_{a_\eta}^{p_\eta}(\eta)$

3D $\quad N_a(\xi,\eta,\zeta) = l_{a_\xi}^{p_\xi}(\xi)\; l_{a_\eta}^{p_\eta}(\eta)\; l_{a_\zeta}^{p_\zeta}(\zeta)$

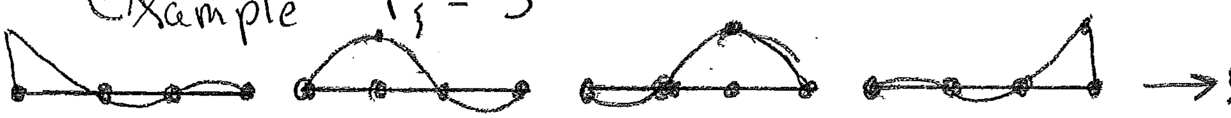
$p_\xi$ = polynomial order in $\xi$ direction

$a_\xi$ - $\xi$ direction "station" for node a



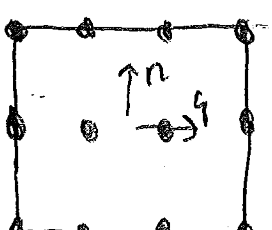

(Same for other coordinates

example $\quad p_\xi = 3$



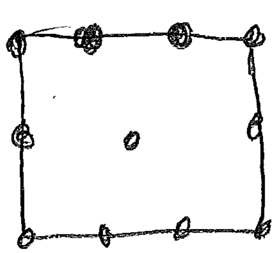

$l_1^3(\xi) \qquad l_2^3(\xi) \qquad l_3^3(\xi) \qquad l_4^3(\xi)$

Use of this formula requires the same order polynomial in a specific direction at each station in the other directions




Cubic in $\xi$
quadratic in $\eta$

Lagrangian Element

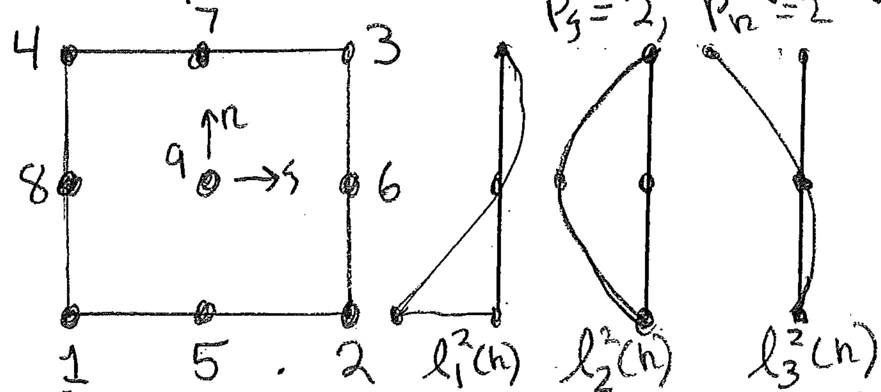


$\leftarrow$ not cubic in $\xi$

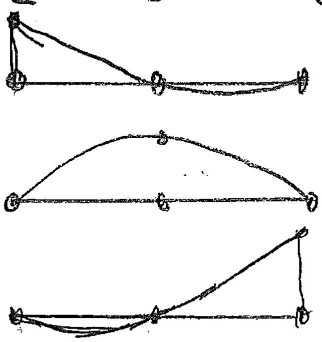
Not a Lagrangian Element

# Example - 9 noded Lagrangian Quadratic

$P_\xi = 2, \; P_\eta = 2$




$$\ell_1^2(\eta) = \tfrac{1}{2}\eta(\eta-1)$$
$$\ell_2^2(\eta) = 1-\eta^2$$
$$\ell_3^2(\eta) = \tfrac{1}{2}\eta(\eta+1)$$



$$\ell_1^2(\xi) = \tfrac{1}{2}\xi(\xi-1)$$
$$\ell_2^2(\xi) = 1-\xi^2$$
$$\ell_3^2(\xi) = \tfrac{1}{2}\xi(\xi+1)$$

$N_1 = \ell_1^2(\xi)\,\ell_1^2(\eta)$    $N_4 = \ell_1^2(\xi)\,\ell_3^2(\eta)$    $N_7 = \ell_2^2(\xi)\,\ell_3^2(\eta)$

$N_2 = \ell_3^2(\xi)\,\ell_1^2(\eta)$    $N_5 = \ell_2^2(\xi)\,\ell_1^2(\eta)$    $N_8 = \ell_1^2(\xi)\,\ell_2^2(\eta)$

$N_3 = \ell_3^2(\xi)\,\ell_3^2(\eta)$    $N_6 = \ell_3^2(\xi)\,\ell_2^2(\eta)$    $N_9 = \ell_2^2(\xi)\,\ell_2^2(\eta)$

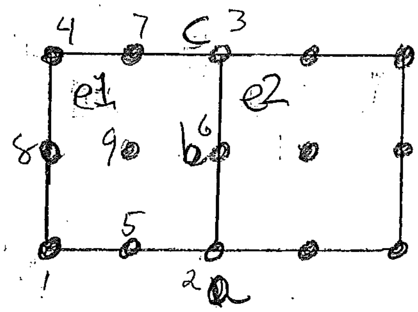
## Conditions C1-C3 ($m=1$ case)

C1 - Intraelement continuity <u>need $C^1$</u>

In $\xi$-$\eta$, its fine | In x-y also need positive $|J|$ in mapping - more later

C2 - Inter element continuity - need $C^0$

Need to show $u^h$ is continuous between elements - Will again look at u on boundary between elements - If uniquely defined in terms of nodal values on the boundary - $u^h$ is continuous since the nodal values are shared.




Since elements e1 and e2 share dof at nodes a, b, c want $u^h$ on edge to be a quadratic functions defined by $u_a^h$, $u_b^h$ and $u_c^h$

$$u^h = \sum_{a=1}^{9} N_a d_a = \sum_{a=1}^{9} N_a u_a^h$$

Consider element e1 – along the edge a-b-c value of $\xi = 1$; thus

$$\ell_1^2(\xi) = \ell_2^2(\xi) = 0 \quad \text{and} \quad \ell_3^2(\xi) \equiv 1$$

Thus

$$u^h\Big|_{a-b-c} = 0\,u_1^{e1} + N_2\Big|_{\xi=1} \overset{u_2^{e1}}{\overset{\|}{u_a^h}} + N_3\Big|_{\xi=1} \overset{u_3^{e1}}{\overset{\|}{u_c^h}} + 0\,u_4^{e1} + 0\,u_5^{e1}$$
$$+ N_6\Big|_{\xi=1} \underset{u_6^{e1}}{\underset{\|}{u_b^h}} + 0\,u_7^{e1} + 0\,u_8^{e1} + 0\,u_9^{e1}$$

$$u^h\Big|_{a-b-c} = \tfrac{1}{2}\eta(\eta-1)\,u_a^h + \tfrac{1}{2}\eta(\eta+1)\,u_c^h + (1-\eta^2)\,u_b^h$$

A quadratic function uniquely defined in terms of the three nodal values on that edge that are shared by the two elements – the $C^0$ inter element continuity requirement met (after also checking the other edges)

C3 - Completeness

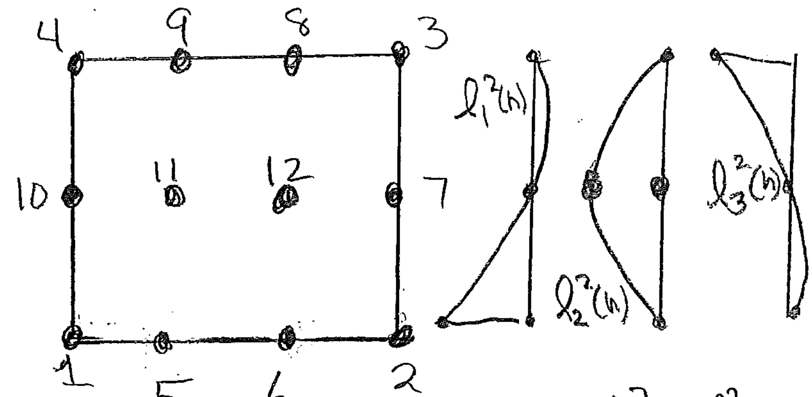
Since these are interpolating shape functions we can use the result we had for that case which says you are set if:

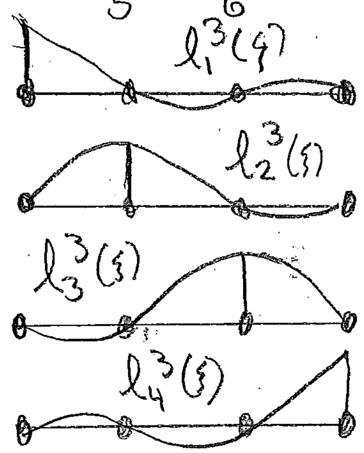
$$\sum N_a = 1, \quad \sum N_a x_a = x \text{ and } \sum N_a y_a = y$$

Start by checking $\sum N_a = 1$

$\sum_{a=1}^{9} N_a = 1$ is met ← check yourself

If you select "isoparametric mappings - that is $\sum N_a \underset{\sim}{x}_a = \underset{\sim}{x}$ you will be set.




$N_1 = l_1^3(\xi)\, l_1^2(\eta)$ $\quad N_7 = l_4^3(\xi)\, l_2^2(\eta)$

$N_2 = l_4^3(\xi)\, l_1^2(\eta)$ $\quad N_8 = l_3^3(\xi)\, l_3^2(\eta)$

$N_3 = l_4^3(\xi)\, l_3^2(\eta)$ $\quad N_9 = l_2^3(\xi)\, l_3^2(\eta)$

$N_4 = l_1^3(\xi)\, l_3^2(\eta)$ $\quad N_{10} = l_1^3(\xi)\, l_2^2(\eta)$

$N_5 = l_2^3(\xi)\, l_1^2(\eta)$ $\quad N_{11} = l_2^3(\xi)\, l_2^2 \eta$

$N_6 = l_3^3(\xi)\, l_1^2(\eta)$ $\quad N_{12} = l_3^3(\xi)\, l_2^2(\eta)$

Conditions C1-C3 ← same stuff as before

Lagrangian Elements

$$1$$

L-L

$$\xi \quad \eta$$

Q-Q

$$\xi^2 \quad \xi\eta \quad \eta^2$$

C-C

Q-C

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$$

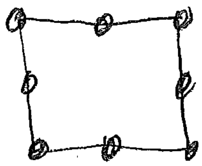
$$\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5$$

$$\xi^6 \quad \xi^5\eta \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \xi\eta^5 \quad \eta^6$$

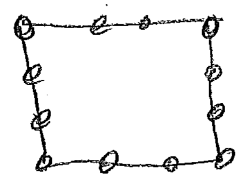
$$\xi^7 \quad \xi^6\eta \quad \xi^5\eta^2 \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi^2\eta^5 \quad \xi\eta^6 \quad \eta^7$$

# Serendipity element, variable # of nodes per edge, and other options

Serendipity – typically nodes on edges only

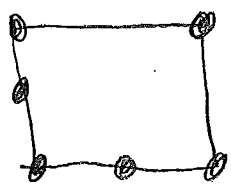


8 noded most common



12 noded

Variable # of nodes –



Useful for transition of element orders

Can not just apply products of Lagrange polynomials since those will have "zeros" that will not be accounted for – not enough conditions.

Ad-hoc construction methods required

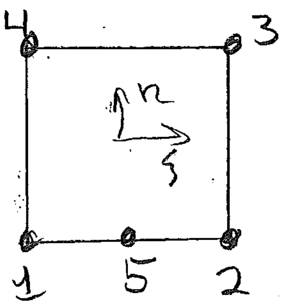
Typical procedure

1. Start with "minimal" Lagrangian element – typically will be bi-linear
2. Add proper shape functions for added nodes
3. Correct any previous shape functions to satisfy the $N_a(\xi_b) = \delta_{ab}$ condition.

easiest to simply work through examples.

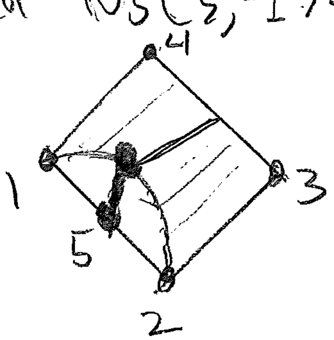
$\hat{N}_a$ - starting "minimal shape functions
$N_a$ - the shape function when done

Start with easest example - an extra node



$\hat{N}_a$, $a=1$-$4$ The bi-linears.
Step. 1 $\hat{N}_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$ $a=1(1)4$

Step 2 - Define a shape function for node 5 that satisfies $N_a(\xi_b,\eta_b) = \delta_{ab}$
need function that is:
1 at node 5, 0 at nodes 1-4
- to be 1 at node 5 and 0 at nodes 1 and 2 need a quadratic in $\xi$. Note that $N_5\Big|_{3-4} = 0$ is required so in the $\eta$ direction
Need $N_5(\xi,-1) \neq 0$, $N_5(\xi,1) = 0$



$$N_5 = \frac{1}{2}\underbrace{\overbrace{(1-\xi^2)}^{l_2^2(\xi)}(1-\eta)}_{l_1^1(\eta)}$$

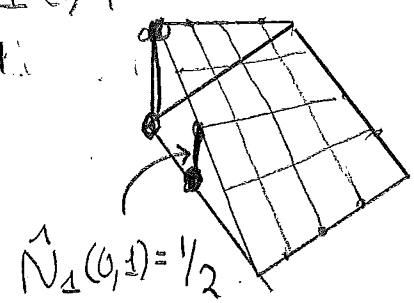
Step. 3 correct any of the $\hat{N}_a$ shape functions that are not zero at the nodes of shape functions defined in step 2

Look at $\hat{N}_a(\xi_5, \eta_5)$ $a = 1(1)4$

$\hat{N}_a(0,-1)$ $a=1(1)4$

$\hat{N}_1(0,-1) = 1/2$

$\hat{N}_a(0,-1) = 0$, $a = 2(1)4$




Easy to correct

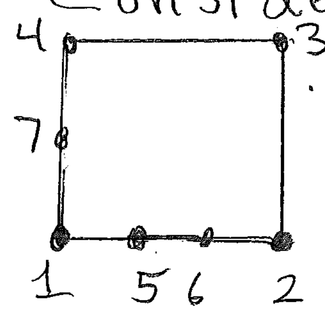
$$N_1 = \hat{N}_1 - \hat{N}_1(0,-1)\, N_5 = \hat{N}_1 - \tfrac{1}{2} N_5$$

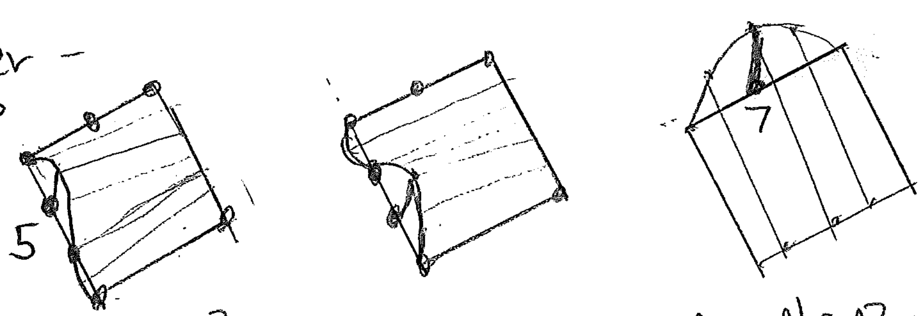
because $N_5 = 0$ at others – causes no problem

We end up with

$N_1 = \hat{N}_1 - 1/2\, N_5$; $N_2 = \hat{N}_2 + 1/2\, N_5$, $N_3 = \hat{N}_3$, $N_4 = \hat{N}_4$

Consider –

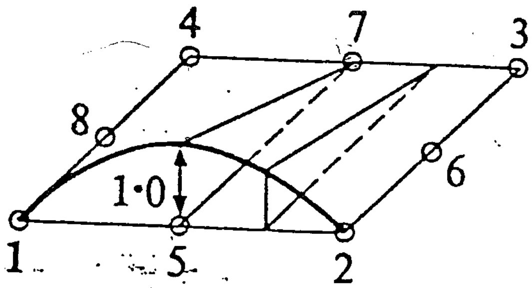



$N_5 = l_2^3(\xi)\, l_1^1(\eta)$ $N_6 = l_3^3(\xi)\, l_1^1(\eta)$ $N_7 = l_1^1(\xi)\, l_2^2(\eta)$

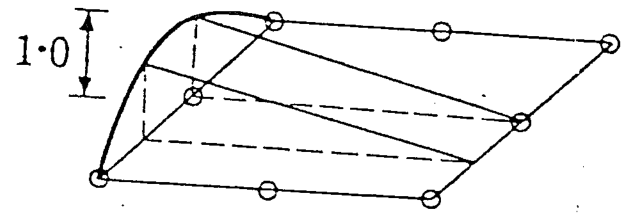
$N_1 = \hat{N}_1 - \tfrac{2}{3} N_5 - \tfrac{1}{3} N_6 - \tfrac{1}{2} N_5$, $N_2 = \hat{N}_2 - \tfrac{1}{3} N_5 - \tfrac{2}{3} N_6$

$N_3 = \hat{N}_3$ , $N_4 = \hat{N}_4 - \tfrac{1}{2} N_7$




(a) $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$

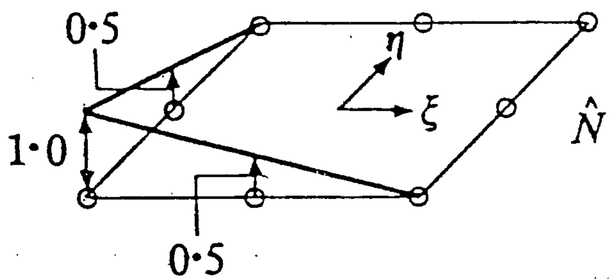
$\ell_2^2(\xi)\,\ell_1^1(\eta)$




(b) $N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$

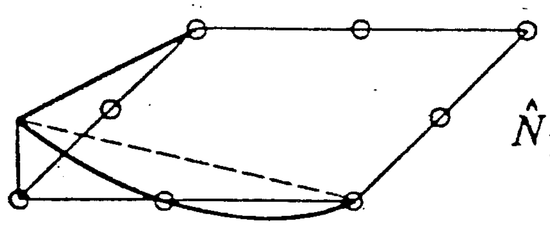
$\ell_1^1(\xi)\,\ell_2^2(\eta)$

Step 1




$\hat{N}_1 = (1-\xi)(1-\eta)/4$

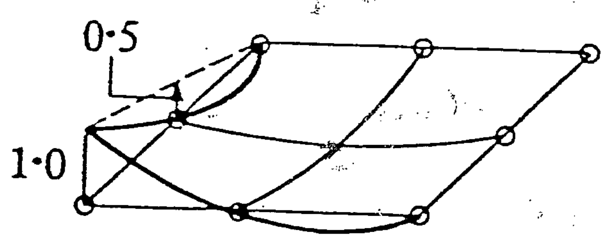
Step 2



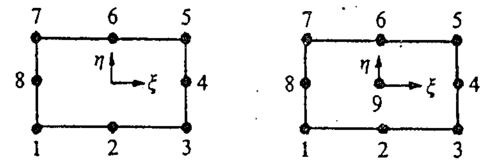
$\hat{N}_1 - \frac{1}{2} N_5$

(c)

Step 3




$N_1 = \hat{N}_1 - \frac{1}{2} N_5 - \frac{1}{2} N_8$

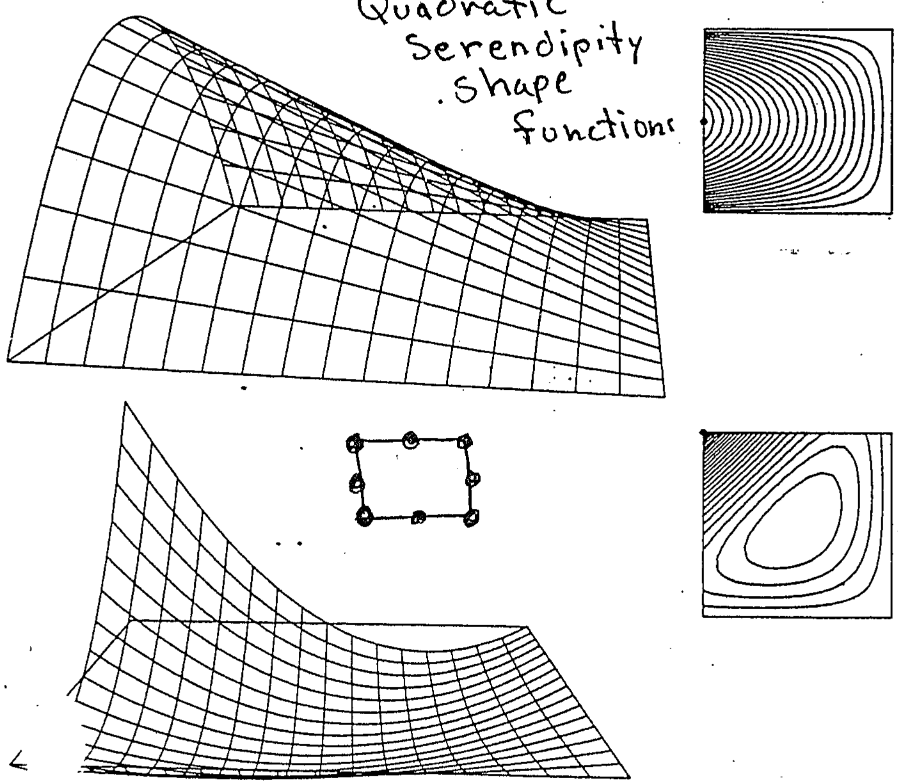



8-node Serendipity element　　9-node Lagrangian element

| Local node number | $\xi_i$ | $\eta_i$ |
|---|---|---|
| 1 | −1 | −1 |
| 2 | 0 | −1 |
| 3 | 1 | −1 |
| 4 | 1 | 0 |
| 5 | 1 | 1 |
| 6 | 0 | 1 |
| 7 | −1 | 1 |
| 8 | −1 | 0 |
| 9 | 0 | 0 |

Fig. 8.7 Quadratic 8 and 9–node rectangular elements.

Quadratic Serendipity shape functions





Quadratic Lagrangian shape functions

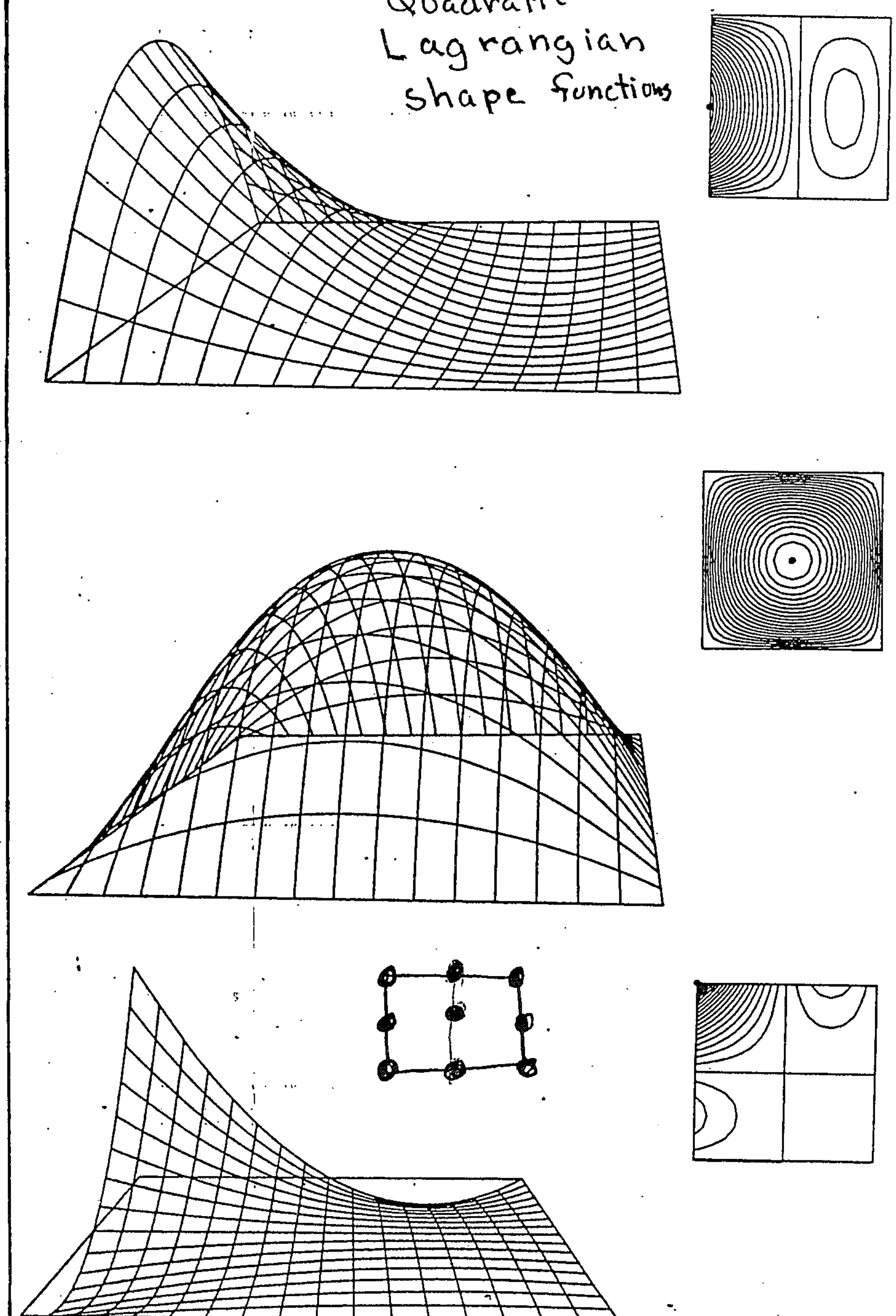


Fig. 8.10 Typical shape functions for the 9-node rectangular element.

III-23

# Serendipity Elements

$$1$$

$$\xi \quad \eta$$

$$\xi^2 \quad \xi\eta \quad \eta^2$$

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$$

$$\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5$$

$$\xi^6 \quad \xi^5\eta \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \xi\eta^5 \quad \eta^6$$

$$\xi^7 \quad \xi^6\eta \quad \xi^5\eta^2 \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi^2\eta^5 \quad \xi\eta^6 \quad \eta^7$$