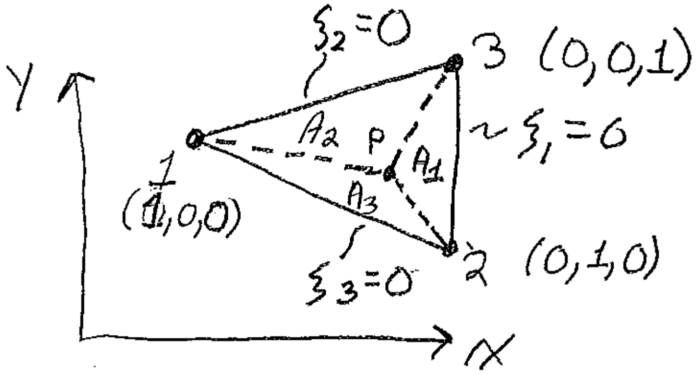
# Area Coordinates and Triangular elements




$$\xi_i = \frac{A_i}{A}$$

A - Area of triangle

$A_i$ - Area of subtriangle $i$

$$\xi_1+\xi_2+\xi_3 = \frac{A_1+A_2+A_3}{A} = \frac{A}{A} = 1$$

Only two independent coordinates -
At some point one need to eliminate one (ex. $\xi_3 = 1-\xi_1-\xi_2$)

Note - The area coordinates on a straight sided element define a convenient linear interpolant

$$u^h = \sum_{a=1}^{3} \xi_a d_a \qquad d_a = u(\underset{\sim}{x}_a)$$

For an isoparametric mapping

$$\underset{\sim}{x} = \sum_{a=1}^{3} \xi_a \underset{\sim}{x}_a \ , \ x = \sum_{a=1}^{3} \xi_a x_a \quad y = \sum_{a=1}^{3} \xi_a y_a$$

In the case of straight sided triangles the mapping between $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ is linear and easily invertable.

This will not be the case for curved elements

The explicit expressions for straight sided elements are:

$$\xi_1(\underset{\sim}{x}) = \frac{1}{2A}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$\xi_2(\underset{\sim}{x}) = \frac{1}{2A}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$\xi_3(\underset{\sim}{x}) = \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

$$A = 1/2\left[(x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)\right] = \text{Area of tri}$$

We use chain rule to calc. derivatives

$$\frac{\partial}{\partial x} = \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i}\frac{\partial \xi_i}{\partial x}, \quad \frac{\partial}{\partial y} = \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i}\frac{\partial \xi_i}{\partial y}$$

for straight sided easy to fill in using above –

$$\frac{\partial \xi_1}{\partial x} = \frac{1}{2A}(y_2 - y_3), \quad \frac{\partial \xi_2}{\partial x} = \frac{1}{2A}(y_3 - y_1), \quad \frac{\partial \xi_3}{\partial x} = \frac{1}{2A}(y_1 - y_2)$$

$$\frac{\partial \xi_1}{\partial y} = \frac{1}{2A}(x_2 - x_3), \quad \frac{\partial \xi_2}{\partial y} = \frac{1}{2A}(x_3 - x_1), \quad \frac{\partial \xi_3}{\partial y} = \frac{1}{2A}(x_1 - x_2)$$

Note that the use of Area (and Volume in $n_{sd}=3$) coordinates is not limited to straight sided elements. Element edges (and faces) can be curved

In the general curved case we again note that we will use a map of $\underset{\sim}{x}$ in terms of $\underset{\sim}{\xi}$

recalling the only two of the area coordinates are independent the following is written assuming you selected $\xi_1$ and $\xi_2$ as the independent coordinates and every where there is a $\xi_3$ you replace it with

$$\xi_3 = 1 - \xi_1 - \xi_2$$

With this we apply chain rule knowing we can take $\partial/\partial\xi_1$ and $\partial/\partial\xi_2$ of things and invert what we have to to produce

$$\partial/\partial x = \frac{1}{|J|}\left(\frac{\partial y}{\partial \xi_2}\frac{\partial}{\partial \xi_1} - \frac{\partial y}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\right)$$

$$\partial/\partial y = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \xi_2}\frac{\partial}{\partial \xi_1} + \frac{\partial x}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\right)$$

$$|J| = \frac{\partial x}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} - \frac{\partial y}{\partial \xi_1}\frac{\partial x}{\partial \xi_2}$$

We also have integration of $\int_{\Omega^e} f(\xi_1,\xi_2,\xi_3)\,d\Omega$ again using $\xi_3 = 1-\xi_1-\xi_2$ we have

$$\int_{\Omega^e} f(\xi_1,\xi_2)\,d\Omega = \int_{\Delta} f(\xi_1,\xi_2)\,|J|\,d\xi_1\,d\xi_2$$

parametric triangle

In the case of straight sided elements the expressions are easier

$$|J| = 2A$$

and integrals of $\xi_1^{\alpha} \xi_2^{\beta}$ have closed form expressions

$$\int_{\Omega^e} \xi_1^{\alpha} \xi_2^{\beta} \, d\Omega = 2A \, \frac{\alpha! \, \beta!}{(\alpha+\beta+2)!}$$

The selection of polynomial terms for triangles will naturally follow the Pascal's triangle -

The interpolating triangles are -

3- nodes → linear

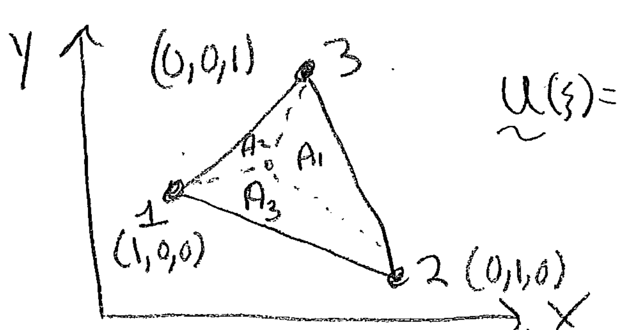
6- nodes → quadratic

10 - nodes → cubic

etc.

3-noded linear, isoparametric triangle

→ Directly interpolate using the area coordinates $N_a = \xi_a = \frac{A_a}{A}$ $a = 1(2)3$




$$\underset{\sim}{u}(\xi) = \sum_{a=1}^{3} N_a(\xi) \, \underset{\sim}{u}_a$$

$$\underset{\sim}{X}(\xi) = \sum_{a=1}^{3} N_a(\xi) \, \underset{\sim}{X}_a$$

$N_1 = \xi_1$ , $N_2 = \xi_2$ , $N_3 = \xi_3 = 1 - \xi_1 - \xi_2$

T-5

$$1$$

$$\xi \quad \eta$$

$$\xi^2 \quad \xi\eta \quad \eta^2$$

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$$

$$\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5$$

$$\xi^6 \quad \xi^5\eta \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \xi\eta^5 \quad \eta^6$$

$$\xi^7 \quad \xi^6\eta \quad \xi^5\eta^2 \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi^2\eta^5 \quad \xi\eta^6 \quad \eta^7$$

C1 - Intra element continuity -
need $u, u_{,x}, u_{,y}$ continuous. ($C^1$)
we have $u(\underset{\sim}{\xi})$, $u_{,\xi_1}(\underset{\sim}{\xi})$ and $u_{,\xi_2}(\underset{\sim}{\xi})$ continuous
by the previous expressions on
$\partial/\partial x$ and $\partial/\partial y$ interms of $\underset{\sim}{\xi}$ we
see so long as $|J| \neq 0$ it is defined
note since we expect positive
area we want $|J| > 0$ in which case
we are fine

C2 - Inter element Continuity
need $u$ to be continuous between elements ($C^0$)

We will meet this of $u$ varies linearly
on each element edge and is defined in terms
of the nodal values for the nodes at
the ends of the edge

On Side 1-2 $\xi_3 = 0$ so $\xi_2 = 1 - \xi_1$

$u|_{1-2} = \xi_1 d_1 + \xi_2 d_2 + \xi_3 d_3 = \xi_1 d_1 + (1-\xi_1) d_2$
varies linearly from $d_1$ to $d_2$ - OK

Side 2-3, $\xi_1 = 0$, $\xi_3 = 1 - \xi_2$
$u|_{2-3} = \xi_1 d_1 + \xi_2 d_2 + \xi_3 d_3 = \xi_2 d_2 + (1-\xi_2) d_3$ - OK

Side 3-1, $\xi_2 = 0$, $\xi_3 = 1 - \xi_1$
$u|_{3-1} = \xi_1 d_1 + \xi_2 d_2 + \xi_3 d_3 = \xi_1 d_1 + (1-\xi_1) d_3$ - OK

C3 – Completeness WRT representing a linear function $(c_0 + c_1 x + c_2 y)$

We already showed that for using interpolating shape functions and isoparametric mappings this reduced to requiring $\sum_{a=1}^{n_{en}} N_a = 1$

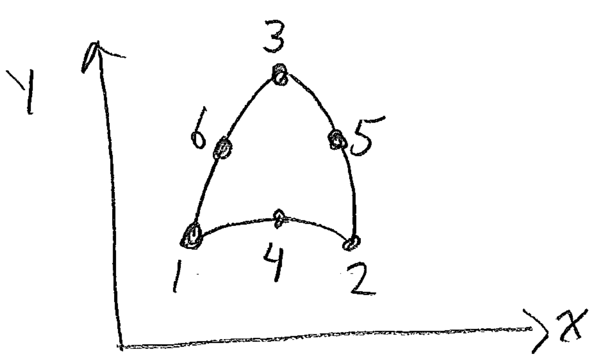
for this case we have

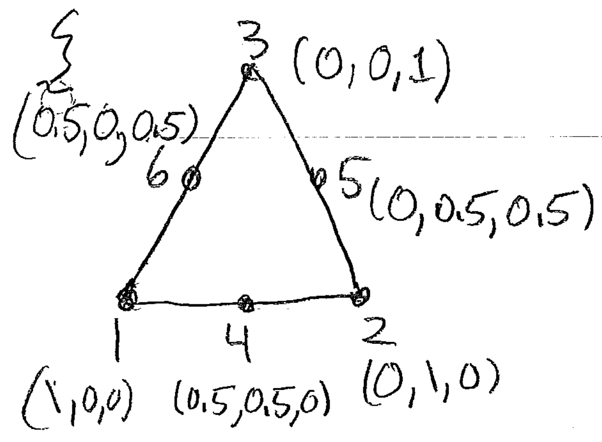
$$\sum_{a=1}^{3} N_a = \xi_1 + \xi_2 + \xi_3 = \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = \frac{A_1 + A_2 + A_3}{A}$$

$$= \frac{A}{A} = \underline{1} \qquad \text{OK}$$

What about higher order triangles

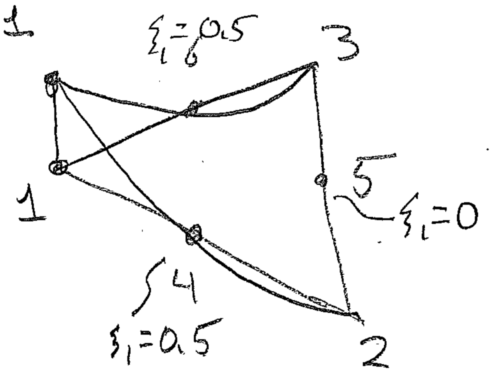
Lets try quadratic isoparametric



Lets consider node 1

$N_1$




Want it a function of $\xi_1$ only – other wise will screw up at some mid side nodes

$\rightarrow N_1 \Leftarrow$ a quadratic in $\xi_1$

Three conditions – at $\xi_1 = 1$, $N_1 = 1$

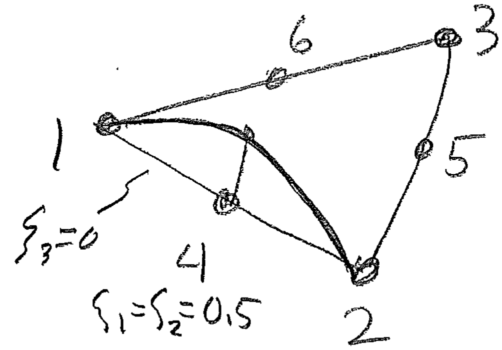
at $\xi_1 = 0.5$, $N_1 = 0$

at $\xi_1 = 0$, $N_1 = 0$

The simple lagrange 1-D quadratic gives us this

$$N_1 = \xi_1 (2\xi_1 - 1)$$
$$N_2 = \xi_2 (2\xi_2 - 1)$$
$$N_3 = \xi_3 (2\xi_3 - 1)$$

What about $N_4, N_5, N_6$




Need a quadratic term that is a function of $\xi_1$ & $\xi_2$ that meets

$N_4(1,0,0) = 0$

$N_4(0.5, 0.5, 0) = 1$

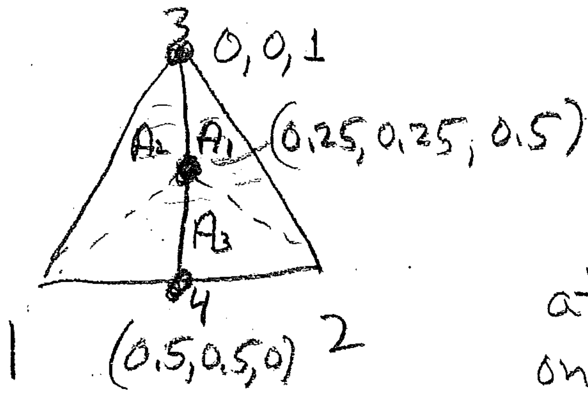
$N_4(0,1,0) = 0$

$$N_4 = 4\xi_1 \xi_2$$
$$N_5 = 4\xi_2 \xi_3$$
$$N_6 = 4\xi_3 \xi_1$$

How does $N_4 = 4\xi_1\xi_2$ vary along the line from node 4 to node 3?

At most is quadratic –
But is it linear?

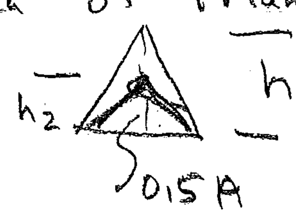
Examine point ½ way along line – see if on the straight line – If not quadratic, If yes linear since the quadratic can not cross a line more than two times –




On Line 4-3
$\xi_1 = \xi_2$

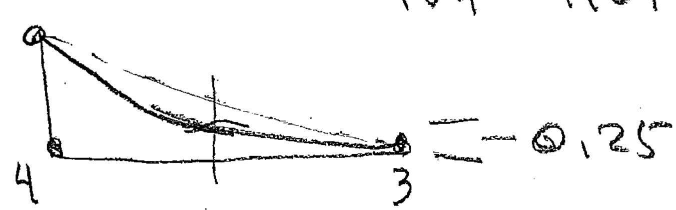
at all time $\xi_3 = 1 - \xi_1 - \xi_2$
on line 4-3 $\xi_3 = 1 - 2\xi_1$

note (0,25, 0,25, 0,5) is ½ way along line (Area of triangle ½ bh)




$N_4$ @ (0,25, 0,25, 0,5)

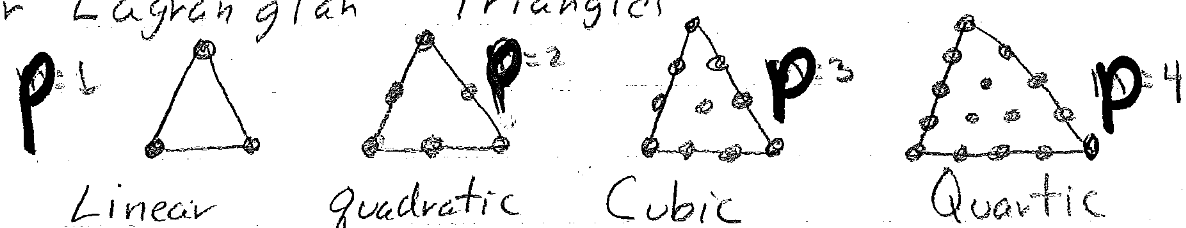
$N_4 = 4(0{,}25)(0{,}25) = 0{,}25$



# Interpolation for triangles

Note: Expressions differ in notation from that in Appendix 3.I, result the same

For Lagrangian triangles




complete polynomials of order $p$

have $n = p+1$ nodes on each side

$$u = \sum_{a=1}^{n_{en}} N_a d_a \qquad n_{en} = \tfrac{1}{2}(p+1)(p+2)$$

\# of terms in complete $p^{th}$ order polynomial

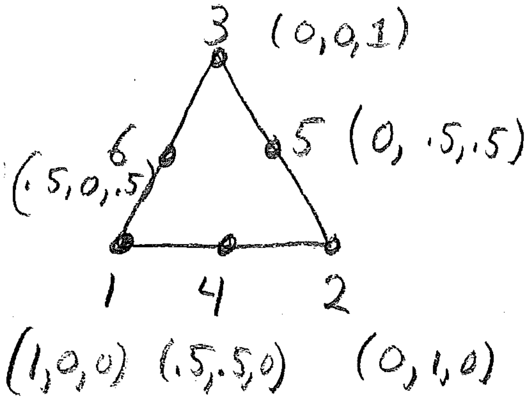
$$N_a(\underset{\sim}{\xi}) = \prod_{i=1}^{3} N_i^a(\xi_i) = N_1^a(\xi_1)\, N_2^a(\xi_2)\, N_3^a(\xi_3)$$

$$N_i^a(\xi_i) = \begin{cases} \displaystyle\prod_{J=1}^{I_i^a} \left( \frac{p\,\xi_i - J + 1}{J} \right) & \text{for } I_i^a \geq 1 \\ 1 & \text{for } I_i^a = 0 \end{cases}$$

$$I_i^a = p\, \xi_i \Big|_{\underset{\sim}{\xi} = \underset{\sim}{\xi}_a}$$

$\Leftarrow$ Second term is the $\xi_i$ coordinate of node a

$p=2$




$$N_1 = N_1^1(\xi_1)\,N_2^1(\xi_2)\,N_3^1(\xi_3)$$

$$I_1^1 = 2(1) = 2,\quad I_2^1 = 2(0) = 0,\quad I_3^1 = 2(0) = 0$$

$$N_2^1 = 1,\quad N_3^1 = 1,\quad N_1^1 = \prod_{J=1}^{2}\left(\frac{2\xi_1 - J + 1}{J}\right) = 2\xi_1\left(\frac{2\xi_1 - 1}{2}\right) = \xi_1(2\xi_1 - 1)$$

$$N_1 = \xi_1(2\xi_1 - 1)(1)(1) = \xi_1(2\xi_1 - 1)$$

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$$N_2 = N_1^2(\xi_1)\,N_2^2(\xi_2)\,N_3^2(\xi_3),\quad I_1^2 = 0 = I_3^2,\quad I_2^2 = 2$$

$$N_1^2 = 1,\quad N_3^2 = 1,\quad N_2^2 = \prod_{J=1}^{2}\left(\frac{2\xi_2 - J + 1}{J}\right) = \xi_2(2\xi_2 - 1)$$

$$N_2 = (1)\,\xi_2(2\xi_2 - 1)(1) = \xi_2(2\xi_2 - 1)$$

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$$N_4 = N_1^4(\xi_1)\,N_2^4(\xi_2)\,N_3^4(\xi_3),\quad I_1^4 = 2(.5) = 1,\quad I_2^4 = 2(.5) = 1,\quad I_3^4 = 0$$

$$N_1^4 = \prod_{J=1}^{1}\left(\frac{2\xi_1 - J + 1}{J}\right) = 2\xi_1,\quad N_2^4 = 2\xi_2,\quad N_3^4 = 1$$

$$N_4 = 4\xi_1\xi_2$$

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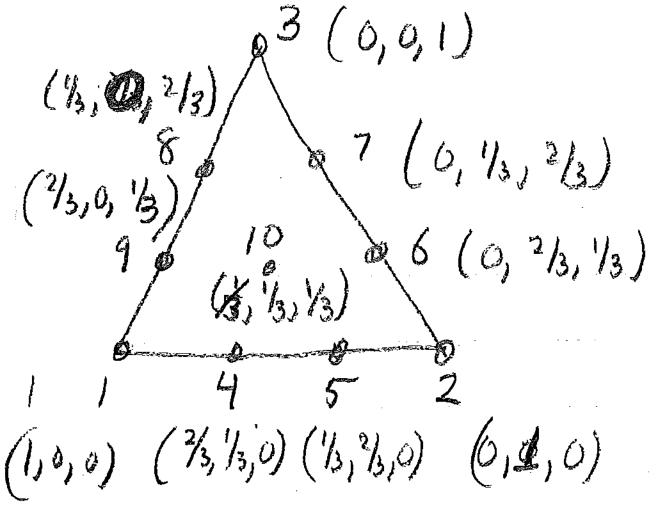


m=3

$p = 3$

$$N_1 = N_1^1(\xi_1)\, N_2^1(\xi_2)\, N_3^1(\xi_3), \quad I_1^1 = 3(1) = 3,\ I_2^1 = 0,\ I_3^1 = 0$$

$$N_1^1 = \prod_{J=1}^{3}\left(\frac{3\xi_1 - J + 1}{J}\right) = 3\xi_1\left(\frac{3\xi_1 - 1}{2}\right)\left(\frac{3\xi_1 - 2}{3}\right) = \frac{\xi_1}{2}(3\xi_1 - 1)(3\xi_1 - 2)$$

$$N_1 = N_1^1\, \cancel{N_2^1}^{\,1}\, \cancel{N_3^1}^{\,1} = \frac{\xi_1}{2}(3\xi_1 - 1)(3\xi_1 - 2)$$

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$$N_4 = N_1^4(\xi_1)\, N_2^4(\xi_2)\, N_3^4(\xi_3), \quad I_1^4 = 3(2/3) = 2,\ I_2^4 = 3(1/3) = 1,\ I_3^4 = 0$$

$$N_1^4 = \prod_{J=1}^{2}\left(\frac{3\xi_1 - J + 1}{J}\right) = 3\xi_1\left(\frac{3\xi_1 - 1}{2}\right) \qquad N_3^4 = 1$$

$$N_2^4 = \prod_{J=1}^{1}\left(\frac{3\xi_2 - J + 1}{J}\right) = 3\xi_2$$

$$N_4 = 3\xi_1\left(\frac{3\xi_1 - 1}{2}\right)3\xi_2 = \frac{9}{2}\xi_1\xi_2(3\xi_1 - 1)$$

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$$N_{10} = N_1^{10}(\xi_1)\, N_2^{10}(\xi_2)\, N_3^{10}(\xi_3), \quad I_i^{10} = 3\left(\frac{1}{3}\right) = 1$$

$$N_i^{10} = \prod_{J=1}^{1}\left(\frac{3\xi_i - J + 1}{J}\right) = 3\xi_i$$

$$N_{10} = 3\xi_1\, 3\xi_2\, 3\xi_3 = 27\,\xi_1\xi_2\xi_3$$