Coverage of Material in Chapter 3

The material we will cover and order it will be covered is as follows:

1. Continuity Requirements – section 3.1
2. Lagrange interpolation – section 3.6
3. Construction of 2-D 4-sided elements (also hexes) – sections 3.2, 3.5, 3.6, 3.7
4. Area coordinates – Appendix 3.I
5. Triangle/tet. Shape functions – Appendix 3.I (not 3.4)
6. Mapping of coordinates – section 3.3
7. Numerical Integration – section 3.8, notes from *Numerical Analysis,* 5th edition Burden and Faires, PWS-Kent Publishing
8. Hermite shape functions – Section 1.16
9. P-version shape functions – notes from *Finite Element Analysis*, Szabo and Babuska, Wiley 1991
10. Isogeometric shape functions – notes from *Isogeometric Analysis: Toward the Integration of CAD and FEA*, Cottrell, Hughes, Bazilevs, Wiley, 2009.

Chapter 3 – Selection of shape functions, natural coordinates, isoparametric elements, numerical integration.

Before discussing the methods to construct shape functions, we want to be sure to understand the conditions that must be met by the shape functions used to represent the trial and weighting functions.

In particular we will cover three requirements the text refers to as C1, C2, C3:

- C1 – Intraelement Continuity – level of continuity within an element
- C2 – Interelement Continuity – level of continuity between elements
- C3 – Completeness – the ability of the function to exactly represent a given order polynomial

Intraelement Continuity Condition C1

Looking at the most demanding term – the energy inner product

$$a^e(w,u) = \int_{\Omega^e} D^n(w) D^m(u)\, d\Omega$$

we see we have derivatives of order n on w and order m on u. Lets assume $m \geq n$ (our most common case is $m = n$). Clearly for our inner product to be meaningful we must have a meaningful integrand, thus we have to be able to take m derivatives. When using polynomials shape functions over elements this will be met by requiring C^m continuity within the element.

Interelement Continuity Condition C2

As already mentioned in Chapter 1, we would like to ensure that we can replace the integral over the entire domain of the sum of element contributions with the sum of element integrals without the need to account for interelement boundary contributions. That is we want:

$$\int_\Omega \sum_{e=1}^{n_{el}} \left(D^n(w) D^m(u) \right) d\Omega = \sum_{e=1}^{n_{el}} \left(\int_{\Omega^e} D^n(w) D^m(u) \, d\Omega \right)$$

Assuming again $m \geq n$, this condition will be satisfied when the jumps in the m^{th} order derivatives between elements are finite (so the integral of them as the boundary thickness goes to zero is zero), This condition is met if our shape functions are C^{m-1} continuous between elements.

This condition will not be hard to meet for our $m = 1$ case since that requires only C^0 interelement continuity. It is also not hard to meet the C^1 (continuous value and continuous first derivative) for the 1D $m = 2$ case. However, meeting the condition of continuous normal slope between elements in 2D and 3D for the $m = 2$ case is quite hard. (As we will discuss at the end of the semester – this makes developing “plate” and “shell” elements for structural mechanics hard.

Completeness Condition C3

The math associated with this condition is complex. However, a simple description is that as the element size approaches zero, the integrand in the energy inner product converges to a constant and that our finite elements must be able to exactly integrate a constant. Taking the common case of $m = n$, this indicates that we need to be able to exactly represent a constant after taking m derivatives. When using polynomial shape functions this indicates we need to be able to exactly represent an m^{th} order polynomial. We will discuss this a bit more below.

For the $m = 1$ case this says we must be able to exactly represent a linear polynomial. For $n_{sd} = 2$

$$u = a_0 + a_1 x + a_2 y$$

For the $m = 2$ case this says we must be able to exactly represent a quadratic polynomial. For $n_{sd} = 2$

$$u = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$

Other Possibly Desirable Conditions to Meet

Completeness of Polynomial Order

Our rate of convergence is a function the highest order polynomial order we complete, not the highest order polynomial terms we may have. View Pascal's triangle – prefer to finish a row before going to the next.

Geometric Isotropy

One would think one wants balance with respect to the polynomial terms – that is of you have x^2 you have y^2.

$$
\begin{array}{c}
1 \\
\xi \quad \eta \\
\xi^2 \quad \xi\eta \quad \eta^2 \\
\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \\
\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4 \\
\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5 \\
\xi^6 \quad \xi^5\eta \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \xi\eta^5 \quad \eta^6 \\
\xi^7 \quad \xi^6\eta \quad \xi^5\eta^2 \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi^2\eta^5 \quad \xi\eta^6 \quad \eta^7
\end{array}
$$

Some more discussion on completeness requirement C3

Consider a 1D FE with $u^h = \sum_{a=1}^{n_{en}} N_a d_a^e$ where N_a are interpolating shape functions, thus $d_a^e = u(x_a)$. For our $m = 1$ case we need to be able to exactly represent:

$$u^h(x) = C_0 + C_1 x$$

Evaluating at the nodes we have

$$u^h(x_a) = C_0 + C_1 x_a = d_a^e$$

putting that back into the FE expansion we have

$$u^h = \sum_{a=1}^{n_{en}} N_a \left(C_0 + C_1 x_a \right)$$

$$u^h = \sum_{a=1}^{n_{en}} N_a + \sum_{a=1}^{n_{en}} N_a C_1 x_a = C_0 \sum_{a=1}^{n_{en}} N_a + C_1 \sum_{a=1}^{n_{en}} N_a x_a$$

if $\sum_{a=1}^{n_{en}} N_a = 1$ and $\sum_{a=1}^{n_{en}} N_a x_a = x$ we get the desired result of

$$u^h(x) = C_0 + C_1 x$$

Note: This result is specifically for the case of interpolating shape functions. If you do not have that, you have to check directly.

Same for $n_{sd} = 2,3$

$$\sum_{a=1}^{n_{en}} N_a = 1 \text{ and } \sum_{a=1}^{n_{en}} N_a \vec{x}_a = \vec{x} = \sum_{a=1}^{n_{en}} N_a x_{ia}, i = 1(1)n_{sd}$$

Consider our 1D linear shape functions $N_1 = \frac{1}{2}(1-\xi)$ and $N_2 = \frac{1}{2}(1+\xi)$. $N_1 + N_2 = 1$. Thus is we select to define the coordinate mapping for the element to be $x(\xi) = \sum_{a=1}^{n_{en}} N_a x_a$, we are sure to be able to exactly represent a linear function (in the case of using interpolating shape functions). Using the same shape functions for the coordinate mapping as is used for the finite element basis is referred as using isoparametric elements.

If the shape functions are not interpolating, you have to perform the algebraic operations needed to show that for $u = a_0 + a_1 x + a_2 y$ in the 2D case.

As mentioned previously we will write our element shape functions in a local element coordinate system and use mappings, such as isoparametric mappings, to account for the actual element shapes.

- Different element topologies will use different local coordinates (for example we will see a difference between quadrilaterals and triangles).
- There are different options for the definition of element coordinate systems on even a single element topology.