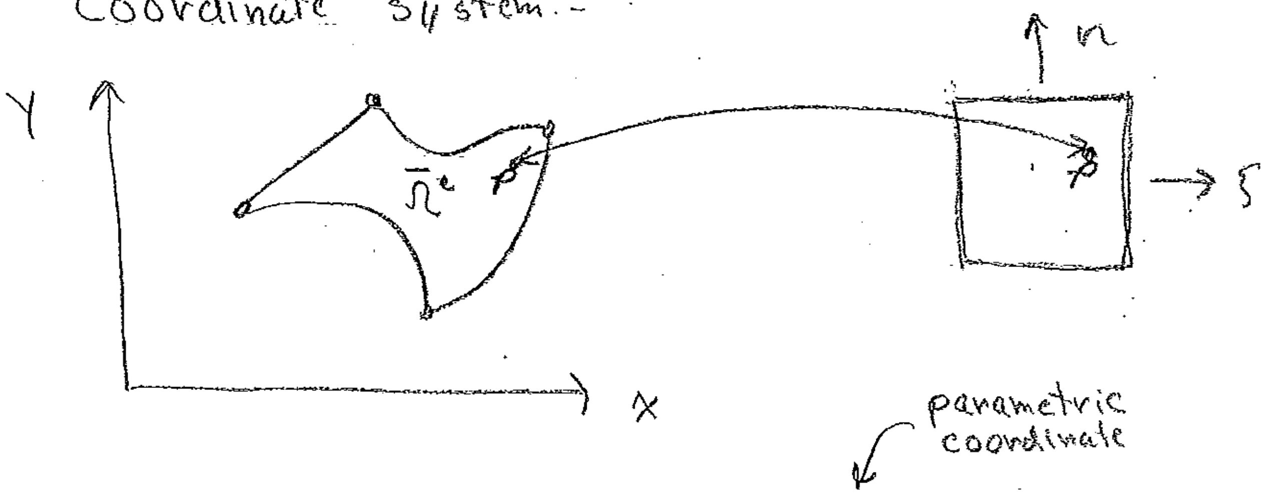
Lets revisit Mappings - a bit more general view

Relating our elements with shape functions defined in a parametric coordinate system with the element in the real (global) coordinate system. -



need maping acting on $\square$ to relate it to global coordinates -

$$\underset{\sim}{x} : \square \longrightarrow \bar{\Omega}^e$$ ← going from parametric to real

We will assume elements are 1 to 1 ← To be discussed later

A useful mapping form would be -

$$\underset{\sim}{x} = \underset{\sim}{Q}^e(\xi, \eta)$$

- Common to define on a component by component basis -

$$x = Q^e_x(\xi, \eta) \quad , \quad y = Q^e_y(\xi, \eta)$$

Now during the process of defining the stiffness matrix we need -

$$\frac{\partial}{\partial x} \, , \, \frac{\partial}{\partial y} \quad \text{and} \quad \int_{\Omega} f(x,y)\,\underbrace{dx\,dy}_{d\Omega}$$

What happens to these, if we have these mappings?

consider first differentiation

can use chain rule to relate derivatives in ξ, η to x, y

Start from what we can write based on what we have

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} = \frac{\partial Q_x^e}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial Q_y^e}{\partial \xi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} = \frac{\partial Q_x^e}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial Q_y^e}{\partial \eta}\frac{\partial}{\partial y}$$

Collecting this together yields

$$\begin{Bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{Bmatrix} = \begin{bmatrix} \partial Q_x^e/\partial\xi & \partial Q_y^e/\partial\xi \\ \partial Q_x^e/\partial\eta & \partial Q_y^e/\partial\eta \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = [J] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix}$$

$$\begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \partial Q_y^e/\partial\eta & -\frac{\partial Q_y^e}{\partial\xi} \\ -\partial Q_x^e/\partial\eta & \partial Q_x^e/\partial\xi \end{bmatrix} \begin{Bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{Bmatrix}$$

$$|J| = \frac{\partial Q_x^e}{\partial \xi}\,\partial Q_y^e/\partial\eta - \partial Q_x^e/\partial\eta\,\partial Q_y^e/\partial\xi$$

Integration -

$$dx\,dy = |J|\,d\xi\,d\eta$$

C ... calc

$\int_{\Omega^e} f$ ...

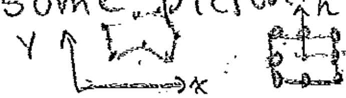
These are some basics – still need to have something to proceed –

Consider for now elements where we use nodal values of the components of $\underset{\sim}{u}$ as unknowns

$$\underset{\sim}{u}^e = \sum_{a=1}^{n_{en}} N_a \underset{\sim}{d}_a$$

$$u_i = \sum_{a=1}^{n_{en}} N_a d_{ia}^e \quad \text{where } d_{ia} = u_i \Big|_{\underset{\sim}{x} = \underset{\sim}{x}_a}$$

Could take the same type of approach – That is assume we define the set of geometry nodes for which we know the coordinates of the nodes in both $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ ← draw some pictures ← next page



define ${}^{g}n_{en}$ as number of geometric nodes

Then

$$\underset{\sim}{x} = \sum_{a=1}^{{}^{g}n_{en}} {}^{g}N_a \underset{\sim}{x}_a^e \quad \text{where } \underset{\sim}{x}_a^e = \underset{\sim}{x}\Big|_{\text{node } a}$$

$$x_i = \sum_{a=1}^{{}^{g}n_{en}} {}^{g}N_a x_{ia}^e$$

The most common choise of ${}^{g}N_a$ is

${}^{g}N_a = N_a$ isoparametric

Define ${}^{g}p$ as polynomial order of ${}^{g}N_a$

" p as polynomial order of N_a

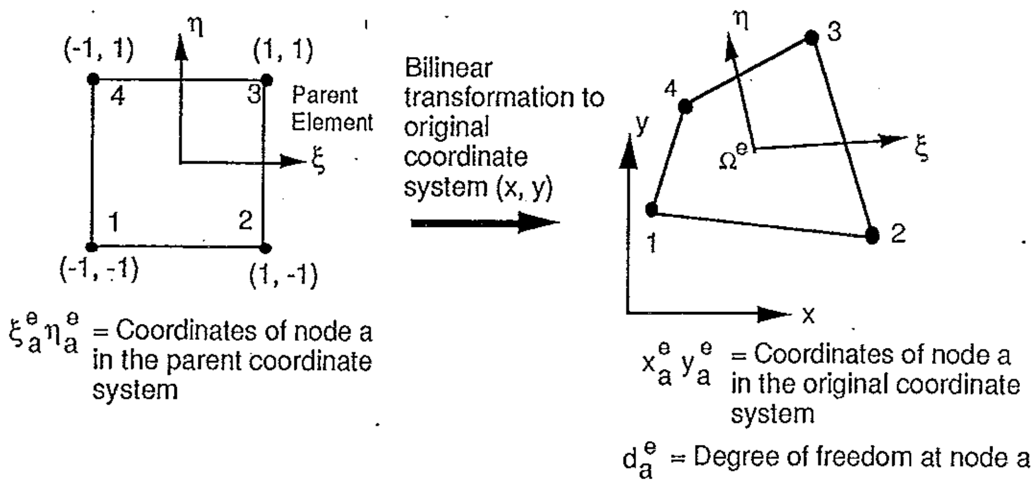
if ${}^{g}p > p$ superparametric element – does not exactly meet constant strain state part of C3

if ${}^{g}p < p$ subparametric – OK with conditions

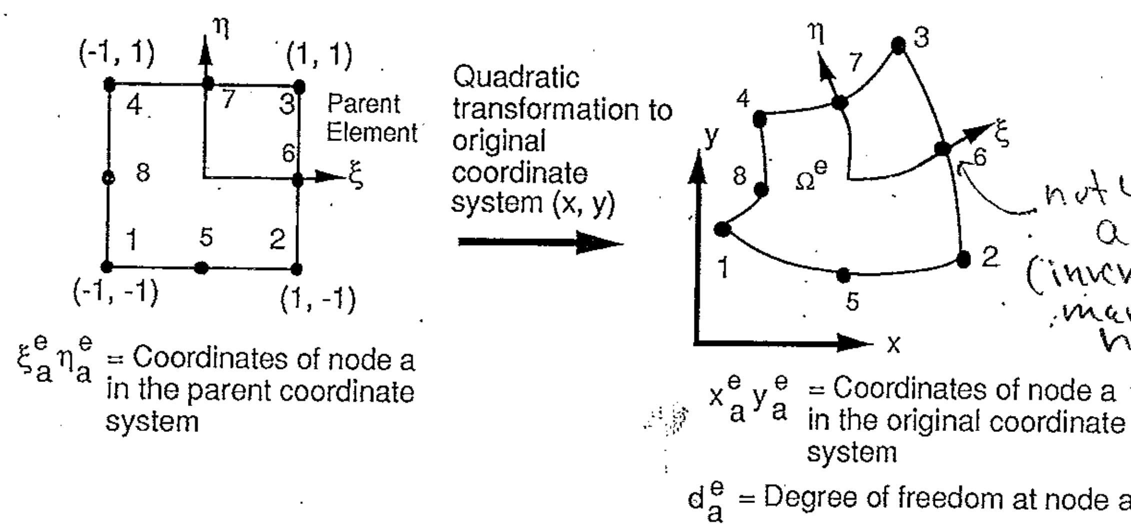
Isoparametric

- These interpolations map the square element in the parent ξ-η system into a more generally shaped element in x-y space whose shape depends on the interpolations, N_a, used.

For example, the linear (bilinear) interpolation maps the square into a quadrilateral with straight sides:



The quadratic (or biquadratic) interpolation maps the square into a quadrilateral with curved (quadratic) sides:



note they are curved (inverse mapping hard! nonlinear

So is the bilinear but can be done

3-D elements add a third coordinate

Lets look again at our intra element continuity condition

Intra element continuity - C1

We require C^m order continuity within the element. We saw it was easy on the parametric element. However, what about accounting for the mapping such that we in sure we can take up to $\partial^m / \partial x_i^m \quad i=1(1)n_{sd}$

To discuss we need to define some terms:

One-to-one: A mapping from parametric to real, $\underset{\sim}{x} : \square \rightarrow \Omega^e \subset \mathbb{R}^{n_{sd}}$, is one-to-one if for each pair of points $\underset{\sim}{\xi}^{(1)}, \underset{\sim}{\xi}^{2} \in \square$ such that $\underset{\sim}{\xi}^{(1)} \neq \underset{\sim}{\xi}^{(2)}$, then

$\underset{\sim}{x}(\underset{\sim}{\xi}^{(1)}) \neq \underset{\sim}{x}(\underset{\sim}{\xi}^{(2)})$ ← Two points do not map to same point in $\bar{\Omega}^e \in \mathbb{R}^{n_{sd}}$

Onto: $\underset{\sim}{x} : \square \rightarrow \bar{\Omega}^e \subset \mathbb{R}^{n_{sd}}$ is onto if $\bar{\Omega}^e = \underset{\sim}{x}(\square)$ which is saying each point in $\bar{\Omega}^e$ is the image of a point in $\square$ under the mapping $\underset{\sim}{x}$.

Differentiable Mapping and Determinate of the Jacobian: We will employ mapping $\underset{\sim}{x}: \square \rightarrow \bar{\Omega}^e \subset \mathbb{R}^{n_{sd}}$ that are differentiable. Thus

we can calculate the determinate of the Jacobian, $|J|$

$n_{sd}=2$ $\quad |J| = \det \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}$

$n_{sd}=3$ $\quad |J| = \det \begin{bmatrix} x_{,\xi} & x_{,\eta} & x_{,\zeta} \\ y_{,\xi} & y_{,\eta} & y_{,\zeta} \\ z_{,\xi} & z_{,\eta} & z_{,\zeta} \end{bmatrix}$

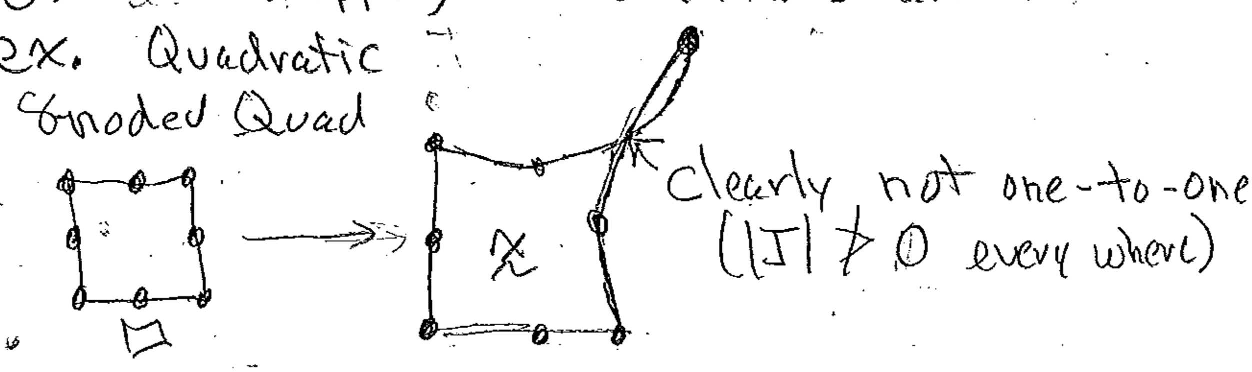
As a consequence of the inverse function theorem (not shown here) that if the mapping $x: \square \rightarrow \bar{\Omega}^e \subset \mathbb{R}^{n_{sd}}$ is

- one-to-one,
- onto,
- C^m, $m \geq 1$, and
- $|J| > 0$ for all $\underset{\sim}{\xi} \in \square$

then the inverse mapping $\underset{\sim}{\xi} = \underset{\sim}{x}^{-1}: \bar{\Omega}^e \rightarrow \square$ exists and is C^m.

Note - A mapping that can meet this requirement must be used <u>and</u> use must ensure to use the mapping so conditions are met.

ex. Quadratic 8noded Quad



note from before for iso parametric

$$\underset{\sim}{x} = \sum_{a=1}^{n_{en}} N_a \underset{\sim}{x}_a$$ and by component

$$x_i = \sum_{a=1}^{n_{en}} N_a x_{ia}^e$$ or in x & y

$$x = \sum_{a=1}^{n_{en}} N_a x_a^e$$

$$y = \sum_{a=1}^{n_{en}} N_a y_a^e$$

Using these last two we can rewrite

$$\frac{\partial}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right) =$$

$$\frac{\partial}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right)$$

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

Lets look at integration a bit closer
consider 2-D

$$\int_{\Omega^e} f(x,y)\, d\Omega = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, |J(\xi,\eta)|\, d\xi\, d\eta$$

More specifically consider the element Stiffness matrix

$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega \quad = \quad \int_{\Omega^e}^{b} N_{a,x}\, D\, N_{B,x}\, dx$$

$\uparrow$ $N_a(\xi)$

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2 \ldots \underset{\sim}{B}_{nen}]$$

for heat conduction

$$\underset{\sim}{B}_a = \underset{\sim}{\nabla} N_a = \begin{Bmatrix} N_{a,x} \\ N_{a,y} \end{Bmatrix} \qquad \underset{\sim}{D} = [k]_{2\times 2}$$

(assume its constant)

for elasticity

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,x} & 0 \\ 0 & N_{a,y} \\ N_{a,y} & N_{a,x} \end{bmatrix}, \qquad \underset{\sim}{D} = [D]_{3\times 3}$$

assume constant

so a typical term in the matrix is

$$\frac{\partial N_a}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial N_a}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial N_a}{\partial \eta}\right) = \frac{1}{|J|}\left(y_{,\eta} N_{a,\xi} - y_{,\xi} N_{a,\eta}\right)$$

$$\frac{\partial N_a}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial N_a}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial N_a}{\partial \eta}\right) = \frac{1}{|J|}\left(-x_{,\eta} N_{a,\xi} + x_{,\xi} N_{a,\eta}\right)$$

substituting this in - gives

for heat conduction

$$\{\underset{\sim}{B_a}\} = \frac{1}{|J|}\begin{Bmatrix} y_{,\eta} N_{a,\xi} - y_{,\xi} N_{a,\eta} \\ -x_{,\eta} N_{a,\xi} + x_{,\xi} N_{a,\eta} \end{Bmatrix} = \frac{1}{|J|}\hat{\underset{\sim}{B_a}}$$

for elasticity

$$\underset{\sim}{B_a} = \frac{1}{|J|}\begin{bmatrix} y_{,\eta} N_{a,\xi} - y_{,\xi} N_{a,\eta} & 0 \\ 0 & -x_{,\eta} N_{a,\xi} + x_{,\xi} N_{a,\eta} \\ -x_{,\eta} N_{a,\xi} + x_{,\xi} N_{a,\eta} & y_{,\eta} N_{a,\xi} - y_{,\xi} N_{a,\eta} \end{bmatrix} = \frac{1}{|J|}\hat{\underset{\sim}{B_a}}$$

$$\underset{\sim}{B} = [\underset{\sim}{B_1}, B_2 \cdots B_{nen}] = \frac{1}{|J|}[\hat{\underset{\sim}{B_1}}, \hat{\underset{\sim}{B_2}} \cdots \hat{\underset{\sim}{B_{nen}}}] = \frac{1}{|J|}\hat{B}$$

substituting this in yields the stiffness integral yields

$$\underset{\sim}{k}^e = \int\int \frac{1}{|J|}\hat{\underset{\sim}{B}}^T \underset{\sim}{D} \frac{1}{|J|}\hat{\underset{\sim}{B}}\, |J|\, d\xi\, d\eta$$

see why $|J| \neq 0$

$$k^e = \int_{-1}^{1}\int_{-1}^{1} \hat{\underset{\sim}{B}}^T \underset{\sim}{D} \hat{\underset{\sim}{B}} \frac{1}{|J|}\, d\xi\, d\eta$$ ⇐ in general this is a rational function

→ when the element shapes are general since $|J|$ is in that case $f(\xi, \eta)$ when N_a has terms greater than linear

- when it is a rational function realize integration is no longer given by simple formula for polynomial integrands — typically integrate approximately

Homework Hint

1-D problem

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} = J(\xi)\frac{\partial}{\partial x}, \quad \underline{\frac{\partial}{\partial x} = \frac{1}{J(\xi)}\frac{\partial}{\partial \xi}}$$

$$dx = J(\xi)\, d\xi$$

$$J(\xi) = \frac{\partial x}{\partial \xi}$$

$$\frac{\partial N_a}{\partial x} = \frac{1}{J(\xi)}\frac{\partial N_a}{\partial \xi}$$