Numerical Integration

References:

- Numerical Analysis, Burden and Faires, Pws-Kent Publishing Co.
- F.E. Procedures in Engineering Analysis, K.J. Bathe, Prentice Hall

We have seen that in many cases the integrand to our element stiffness integral is a rational polynomial. Thus we are not likely to have exact integration formula for the integrals. Thus we must integrate numerically in that case. Even if when we can integrate exactly it is typically more convenient and just as efficient, to use numerical integration.

Numerical quadrature - Look at 1-D

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{\ell=1}^{n_{int}} w_\ell\, g(\xi_\ell) + R_{n_{int}-1}$$

How do we get to such a formula - will use an interpolating polynomial to approximate the integrand and integrate those terms to obtain our weights. Using our Lagrange polynomials

$$g(\xi) = \sum_{\ell=1}^{n_{int}} g(\xi_\ell)\, L_\ell^{n_{int}-1}(\xi) + \vec{R}$$

in which case our integral becomes

$$\int_{-1}^{1} g(\xi)\,d\xi = \int_{-1}^{1} \sum_{\ell=1}^{n_{int}} g(\xi_\ell)\, L_\ell^{n_{int}-1}\, d\xi + \int_{-1}^{1} \hat{R}\, d\xi$$

$$= \sum_{\ell=1}^{n_{int}} \left(g(\xi_\ell) \int_{-1}^{1} L_\ell^{n_{int}-1}\, d\xi \right) + R_{n_{int}-1} = \sum_{\ell=1}^{n_{int}} w_\ell\, g(\xi_\ell) + R_{n_{int}-1}$$

$\Rightarrow$ note $-\ w_\ell = \int_{-1}^{1} L_\ell^{n_{int}-1}\, d\xi$

It can be shown that at some location $-1 \le \hat{\xi} \le 1$

$$\hat{R} = \left(\frac{d^{n_{int}} g(\xi)}{d\xi^{n_{int}}} \Big/ n_{int}! \right) \prod_{\ell=1}^{n_{int}} (\xi - \xi_\ell)$$

$$R_{n_{int}-1} = \int_{-1}^{1} \hat{R}\, d\xi$$

This method is referred to as Newton-Cotes

Two common Newton Cotes rules:

Trapezoidal Rule, $n_{int} = 2$, points at $\xi = -1, \xi = 1$

$$w_1 = \int_{-1}^{1} l_1^1\, d\xi = \int_{-1}^{1} \tfrac{1}{2}(1-\xi)\, d\xi = 1$$

$$w_2 = \int_{-1}^{1} l_2^1\, d\xi = \int_{-1}^{1} \tfrac{1}{2}(1+\xi)\, d\xi = 1$$

$$R = -\tfrac{2}{3}\, g_{\xi\xi}(\hat{\xi})$$ $\Leftarrow$ We will exactly integrate a linear

Simpson's Rule, $n_{int}=3$, points at $\xi=-1, \xi=0, \xi=1$

$$w_1 = \int_{-1}^{1} l_1^2 d\xi = \int_{-1}^{1} \frac{1}{2}(\xi^2-\xi)\, d\xi = 1/3$$

$$w_2 = \int_{-1}^{1} l_2^2 d\xi = 4/3 \;, \quad w_3 = \int_{-1}^{1} l_3^2 d\xi = 1/3$$

The error equation we saw before would say the error is $\alpha\, g_{,\xi\xi\xi\xi}$

However, a more careful analysis shows that the error is:

$$R = \frac{-8 f^{iv}(\xi)}{90}$$ ← proportional to 4th derivative

This says cubics would be integrated exactly

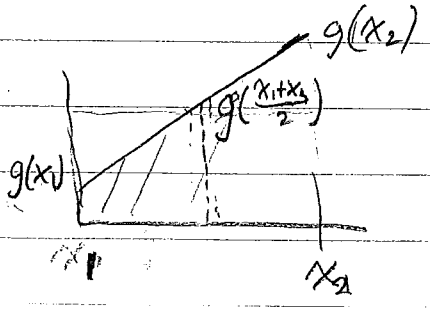
How can that be - We only used 3 conditions (values at 3 points) - So why do we do better than 3-1=2 for the order curve done exactly?

⇒ The positions of the evaluation points in the numerical integration can be used as conditions - In this case one point (the center point) is at such an "optimal" location

There is a procedure to find these points which leads to the Guass Quadrature ~~rules~~ where with n_{int} - integration points you can exactly integrate an order $2n_{int}-1$ order polynomial

⇒ Note - we are not saying we exactly represent it - just exactly integrate it.

Consider a linear curve - Can easily exactly integrate using average value (center value) times interval ⇒ $A = g\left(\frac{x_1+x_2}{2}\right)(x_2-x_1)$



Use Gauss - Legendre Polynomials to find the optimal points to use in deriving the locations for the Gauss integration rules

From before we had the approx. of the integrand $n_{int} = n+1$

$$g(\xi) = P_n(\xi) + \hat{R}_n = \sum_{l=1}^{n_{int}} g(\xi_l) L_l^n(\xi) + \hat{R}_n$$

We want to replace $\hat{R}_n$ with an expression that gives a handle on the other n_{int} conditions

Define - $P(\xi) = (\xi - \xi_1)(\xi - \xi_2) \cdots (\xi - \xi_{n_{int}})$
this is an n_{int} order polynomical
note: $P(\xi_i) = 0 \quad i = 1(1) n_{int}$

now write:

$$g(\xi) = \sum_{l=1}^{n_{int}} g(\xi_l) L_l^{n_{int}-1}(\xi) + P(\xi)(\beta_0 + \beta_1 \xi + \beta_2 \xi^2 + \cdots)$$

($\psi(\xi)$; Taylors series)

$$g(\xi) = \psi(\xi) + \sum_{j=0}^{n_{int}-1} \beta_j \xi^j P(\xi) + \hat{R}_{2n_{int}-1}$$

always zero at every integration point

An order $2n_{int}-1$ order polynomial that still interpulates at the n_{int} integration points

This is the representation we put into the integration

$$\int_{-1}^{1} g(\xi)\,d\xi = \sum_{\ell=1}^{n_{int}} g(\xi_\ell) \int_{-1}^{1} L_\ell^{(n_{int}-1)}(\xi)\,d\xi + \sum_{j=0}^{n_{int}-1} \beta_j \int_{-1}^{1} \xi^j P(\xi)\,d\xi + \int_{-1}^{1} \hat{R}_{2n_{int}-1}\,d\xi$$

Note that in the above expression we do not know the n_{int} ξ's or the n_{int} β_j's

We in fact do not want the β_j's - That is we want to select the locations So the contributions of the β_j's, terms are zero That is done by forcing the multiplier of each β_j to be zero ← the other n_{int} conditions

That is we want -

$$\int_{-1}^{1} \xi^j P(\xi)\,d\xi = 0 \qquad j = 0(1) n_{int}-1$$

This gives n_{int} equations in n_{int} unknowns.

Example - 2 point integration

$j = 0$ $\int_{-1}^{1} \xi^0 (\xi-\xi_1)(\xi-\xi_2)\,d\xi = 0$ yields $\xi_1 \xi_2 = -1/3$

$j = 1$ $\int_{-1}^{1} \xi (\xi-\xi_1)(\xi-\xi_2)\,d\xi = 0$ $\xi_1 + \xi_2 = 0$

$\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$

With the locations determined we can go back to determine the weights to use

$$w_1 = \int_{-1}^{1} \ell_1'(\xi)\, d\xi = \int_{-1}^{1} \frac{\xi - \xi_2}{\xi_1 - \xi_2}\, d\xi = \int_{-1}^{1} \frac{\xi - 1/\sqrt{3}}{-1/\sqrt{3} - 1/\sqrt{3}}\, d\xi = 1$$

$$w_2 = \int_{-1}^{1} \ell_2'(\xi)\, d\xi = \int_{-1}^{1} \frac{\xi - \xi_1}{\xi_2 - \xi_1}\, d\xi = \int_{-1}^{1} \frac{\xi - (-1/\sqrt{3})}{1/\sqrt{3} - (-1/\sqrt{3})}\, d\xi = 1$$

$n_{int} = 1$, $\xi_1 = 0$, $w_1 = 2$, $R = \dfrac{g_{,\xi\xi}(\vec{\xi})}{3}$

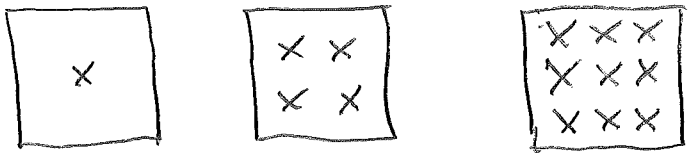
$n_{int} = 2$, $\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$, $w_1 = w_2 = 1$, $R = \dfrac{g_{,\xi\xi\xi\xi}(\vec{\xi})}{135}$

$n_{int} = 3$, $\xi_1 = -\sqrt{3/5}$, $\xi_2 = 0$, $\xi_3 = \sqrt{3/5}$

$w_1 = w_3 = 5/9$, $w_2 = 8/9$, $R = \dfrac{g_{,\xi\xi\xi\xi\xi\xi}(\vec{\xi})}{15,750}$

Integration for $n_{sd} > 1$

For Quads and Hex elements
Use products of 1-D rules



1×1 2×2 3×3

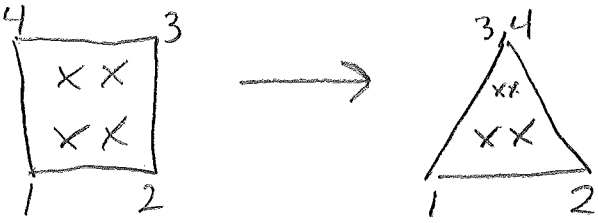
$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1} \sum_{i=1}^{n_{ints}} g(\xi_i,\eta)\,w_i\,d\eta$$

$$= \sum_{j=1}^{n_{intn}} \sum_{i=1}^{n_{ints}} g(\xi_i,\eta_j)\,w_i w_j = \sum_{k=1}^{n_{int}} w_k\, g(\xi_k,\eta_k)$$

$$n_{int} = n_{ints}\, n_{intn}$$

For triangles and tets
A couple of options -

(A) Degenerate product rule

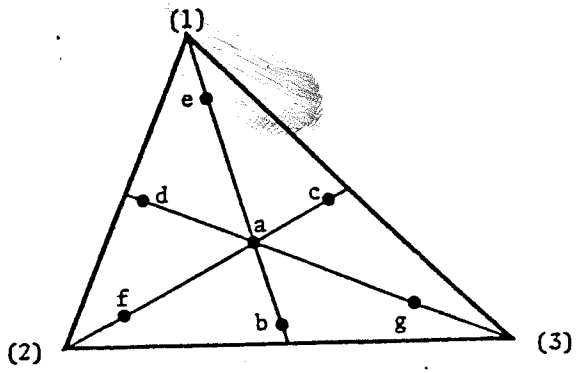


(B) Specifically constructed single loop Simplex integration rules

Hammer's Rule Common for triangles

$$\int_{\Delta} g(\xi_1,\xi_2,\xi_3)\,d\Delta = \sum_{k=1}^{n_{int}} g(\xi_{1k},\xi_{2k},\xi_{3k})\,w_k$$

SEVEN POINT FIFTH ORDER INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	1/3	1/3	1/3	9/80
b	α_1	β_1	β_1	η_1
c	β_1	α_1	β_1	η_1
d	β_1	β_1	α_1	η_1
e	α_2	β_2	β_2	η_2
f	β_2	α_2	β_2	η_2
g	β_2	β_2	α_2	η_2

$\alpha_1 = 0.05971\ 58717\ 89770$

$\beta_1 = 0.47014\ 20641\ 05115$

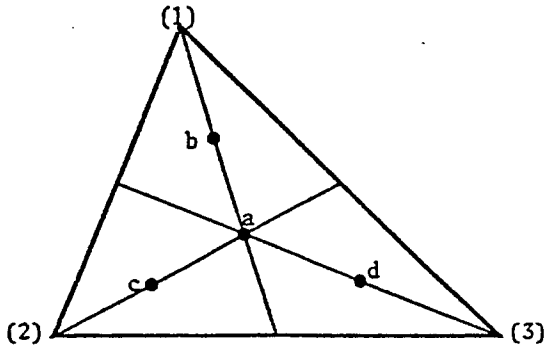
$\eta_1 = 0.06619\ 70763\ 94253$

$\alpha_2 = 0.79742\ 69853\ 53087$

$\beta_2 = 0.10128\ 65073\ 23456$

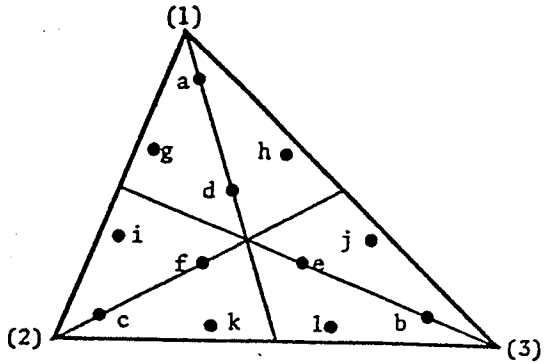
$\eta_2 = 0.06296\ 95902\ 72414$

FOUR POINT CUBIC INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	1/3	1/3	1/3	-27/96
b	3/5	1/5	1/5	25/96
c	1/5	3/5	1/5	25/96
d	1/5	1/5	3/5	25/96

TWELVE POINT SIXTH ORDER INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	α_1	β_1	β_1	η_1
b	β_1	β_1	α_1	η_1
c	β_1	α_1	β_1	η_1
d	α_2	β_2	β_2	η_2
e	β_2	β_2	α_2	η_2
f	β_2	α_2	β_2	η_2
g	α_3	β_3	γ_3	η_3
h	α_3	γ_3	β_3	η_3
i	β_3	α_3	γ_3	η_3
j	β_3	γ_3	α_3	η_3
k	γ_3	α_3	β_3	η_3
l	γ_3	β_3	α_3	η_3

$\alpha_1 = 0.87382\ 19710\ 16996$

$\beta_1 = 0.06308\ 90144\ 91502$

$\eta_1 = 0.02542\ 24531\ 85104$

$\alpha_2 = 0.50142\ 65096\ 58179$

$\beta_2 = 0.24928\ 67451\ 70910$

$\eta_2 = 0.05839\ 31378\ 63190$

$\alpha_3 = 0.63650\ 24991\ 21399$

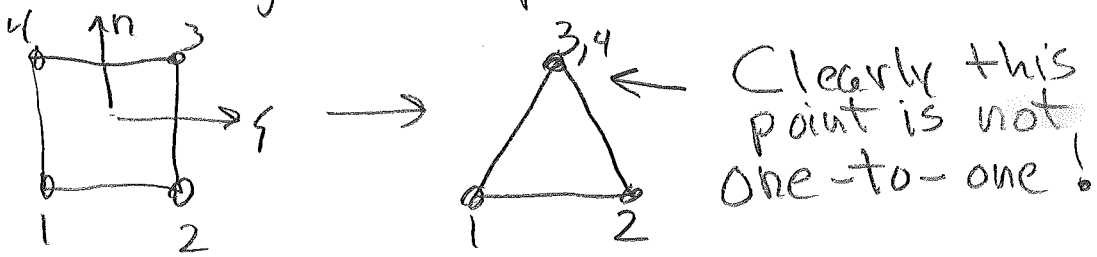
$\beta_3 = 0.31035\ 24510\ 33785$

$\gamma_3 = 0.05314\ 50498\ 44816$

$\eta_3 = 0.04142\ 55378\ 09187$

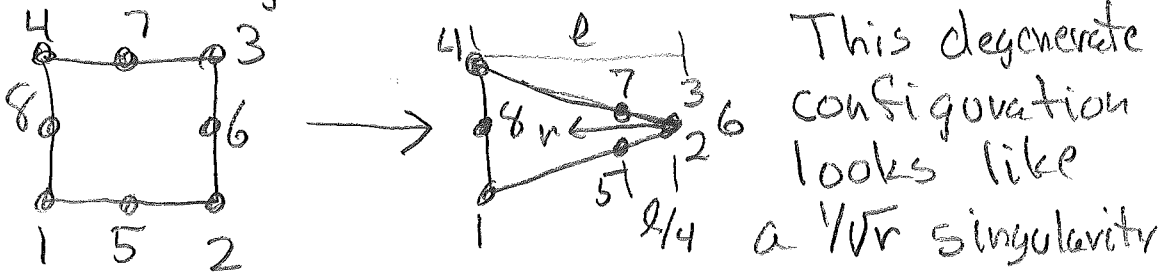
When discussing mappings we discussed the need for $|J|>0$. In the case of numerical integration the $|J|$ is only evaluated at the integration points which are interior to the domain.

The ability to use degenerate elements ($|J|=0$ at vertex) relies on fact that the $|J|$ is evaluated at interior points since $|J|=0$ on the the degenerate portion of the boundary



Clearly this point is not one-to-one!

Sometimes degenerics can be used to advantage



This degenerate configuration looks like a $1/\sqrt{r}$ singularity

Linear fracture mechanics says there is a $1/\sqrt{r}$ strain singularity at the crack tip. Use the above degenerate 8-noded heses and it mimics the $1/\sqrt{r}$ singularity — Used all the time for fracture analyses

What order integration should we use?

Remember for generally shaped elements the integrand is a rational polynomial thus our numerical integration can not be exact. Even when the $|J|$ is constant, which means we would have polynomials (with polynomial shape functions) we may not want to integrate exactly.

So if you do not integrate exactly- How well do you want to integrate?

Take this in two levels

First level - Integrate well enough that our F.E. method will execute and, in the limit, Converge to the exact solution.

Second level - To integrate well enough that we converge at the maximum possible rate as dictated by the highest order complete polynomial in our shape functions.

First Level - Integrate well enough

Two considerations
- to maintain convergence (minimal rate)
- to avoid a singular stiffness matrix

Maintain Convergence (minimal rate)

(min.) Convergence can be maintained so long as we can integrate a constant value of the m^{th} derivative as it appears in the energy inner product correctly

That is we want to correctly integrate

$$\int_{\Omega} c \, d\Omega = \int_{\square} c |J| \, d\square = c \int_{\square} |J| \, d\square$$

Thus we have to integrate the element volume correctly. For polynomial shape functions and isoparametric mappings $|J|$ is a polynomial-

Thus we need to integrate this polynomial exactly to get the correct volume -

This defines a minimum order of integration possible.

To Avoid a Singular Stiffness Matrix

we have to solve $\underset{\sim}{K}\underset{\sim}{d}=\underset{\sim}{F}$

if the stiffness matrix is not full rank then $\underset{\sim}{K}^{-1}$ does not exist and we can not solve the system

If we use too low an order of numerical integration we will not have sufficient conditions and we would find our elements have too many zero eigen values with one or more corresponding to a valid (energy producing) if properly integrated eigenmode)

To look at this lets consider 2-D elasticity and individual element matrices

$\underset{\sim}{k}^{e}_{n_{es}\times n_{ed}}$

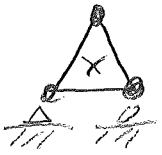
Note since the element has no essential boundary conditions an exactly integrated element stiffness matrix will have 3 zero eigen values, the eigen vectors of which correspond to the 3 rigid body modes

Thus the number of non-zero eigenvalues needed is $n_R = n_{ed} - 3$. We get this if the numerical integration scheme evaluates at least $n_{ed} - 3$ conditions. For elasticity the number of conditions provided

$$n_{cond} = (\#\text{ of strain components})(n_{int})$$

for $n_{sd} = 2$ $\quad = 3 n_{int}$

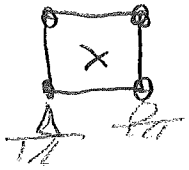
Consider some elements • - node with 2 dof
x - integration point



$n_R = n_{eh} - 3 = 6 - 3 = 3$

$n_{cond} = 3(1) = 3$

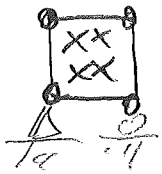
$n_R = n_{cond}$ OK



$n_R = 8 - 3 = 5$, $n_{cond} = 3(1) = 3$

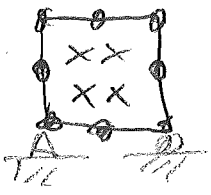
$n_R > n_{cond}$ ⇐ NG

2 zero energy eigenvectors (strains = 0 at integration point)



$n_R = 5$, $n_{cond} = 3(4) = 12$

$n_R < n_{cond}$ (by a lot) OK



$n_R = 16 - 3 = 13$, $n_{cond} = 3(4) = 12$

← strains zero at integration points

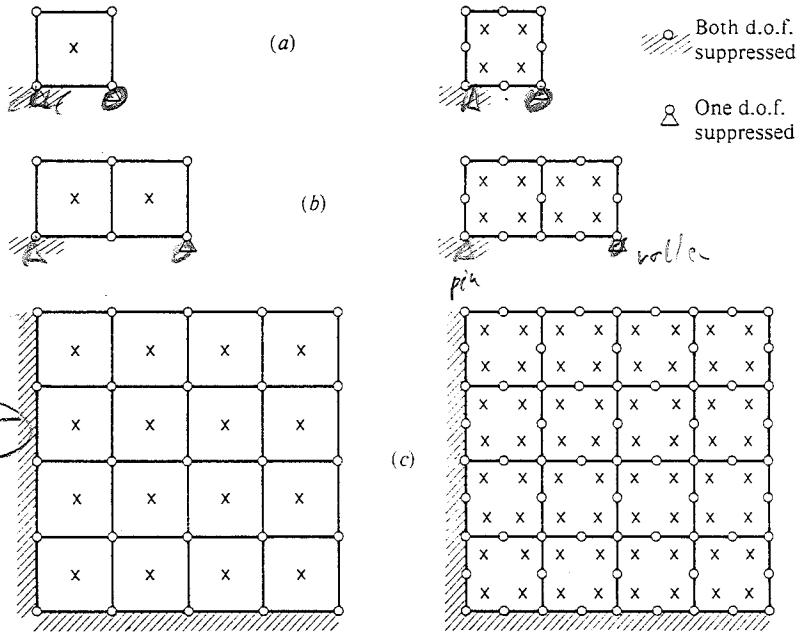
$n_R > n_{cond}$ 1 zero energy eigenvector

Single element with min. essential BC. is singular.

When using multiple elements this mode can not arise and things are fine.

x Integrating point (3 independent relations)

o Nodal point with 2 degrees of freedom



	LINEAR		QUADRATIC	
	Degrees of Freedom	Independent Relation	Degrees of Freedom	Independent Relation
(a)	$4 \times 2 - 3 = 5$	$> 1 \times 3 = 3$ singular	$2 \times 8 - 3 = 13$	$> 4 \times 3 = 12$ singular
(b)	$6 \times 2 - 3 = 9$	$> 2 \times 3 = 6$ singular	$13 \times 2 - 3 = 23$	$< 8 \times 3 = 24$
(c)	$25 \times 2 - 18 = 32$	$< 16 \times 3 = 48$	$48 \times 2 = 96$	$< 64 \times 3 = 192$

Fig. 8.12 Check on matrix singularity in two-dimensional elasticity problems (*a*), (*b*), and (*c*)

To Converge at maximum rate possible

For problems with smooth solutions (no singularities)

$$\|u^h - u\|_m \leq \bar{C} h^{k+1-m} \|u\|_{k+1}$$

m - highest order derivative in $a(w,u)$
k - highest order complete polynomial in u^h

To maintain the convergence rate of $k+1-m$ must use a high enough order integration rule to exactly integrate a polynomial of order

$$(\bar{k} + k - 2m)$$

$\bar{k}$ - highest order monomial term appearing in u^h

 - $\bar{k} = 1$

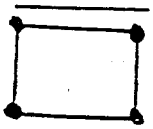
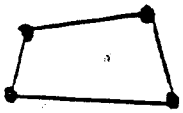
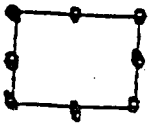
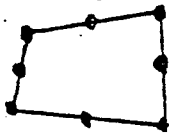
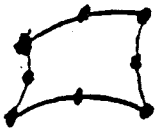
 - $\bar{k} = 2$ ($\xi\eta$ term)

 - $\bar{k} = 2$

 - $\bar{k} = 3$ ($\xi^2\eta$ and $\xi\eta^2$ terms)

 - $\bar{k} = 4$ ($\xi^2\eta^2$ term)

ORDERS OF GAUSSIAN INTEGRATION FOR TWO DIMENSIONAL ELEMENTS

Element	Displacement Model	Geometry Model	Minimum Order [a,b,c]	Usual or Recommended Order [a,d]	Exact or Maximum Order [e]
	Linear	Rectangular	1x1	2x2	2x2
	Linear	Linear	1x1	2x2	3x3
	Quadratic	Rectangular	2x2	2x2	3x3
	Quadratic	Linear	2x2	2x2	3x3
	Quadratic	Quadratic	3x3	3x3	4x4

Notes:

(A) Reference: Zienkiewicz & Hinton, "Reduced Integration, Function Smoothing and Non-Conformity in Finite Element Analysis", Proc. 1976 Int. Conf. on Finite Element Methods in Engineering, Univ. of Adelaide, Australia, and also to be published in the Journal of the Franklin Institute.

(B) Zero energy deformation modes may arise. If these are not interelement compatible, then the recommended order is higher than the minimum order.

(C) Based on volume-integration criterion: Cook, Section 5.7

(D) Reference: Bathe & Wilson, Section 4.7.

(E) All values given are for constant thickness elements.