Hermitian Interpolation

Interpolate value and selected derivatives at selected points.

example – value and derivative at m points

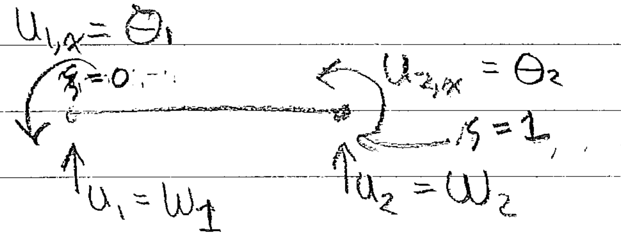
$$u^h = N_1 u_1 + N_2 u_{1,x} + N_3 u_2 + N_4 u_{2,x} + \dots + N_{2m-1} u_m + N_{2m} u_{m,x}$$

example – value and m-1 derivatives at to ends

$$u^h = N_1 u_1 + N_2 u_{1,x} + N_3 u_{1,xx} + \dots + N_m u_{1,(m-1 \text{ derivatives})} + N_{m+1} u_2$$

$$+ N_{m+2} u_{2,x} + \dots + N_{2m} u_{2,(m-1 \text{ derivatives})}$$

Consider a Beam member – value and derivative at each end



assume $x = \xi$ ← unit Length beam to make it simplier

$$u^h = w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2$$

$$u_{,x} = \frac{du^h}{dx} = \theta = N_{1,x} w_1 + N_{2,x} \theta_1 + N_{3,x} w_2 + N_{4,x} \theta_2$$

Four conditions - Need cubic shape functions

$$N_i = a_1^{(i)} + a_2^{(i)}\xi + a_3^{(i)}\xi^2 + a_4^{(i)}\xi^3$$

$$N_{i,\xi} = a_2^{(i)} + 2a_3^{(i)}\xi + 3a_4^{(i)}\xi^2 = N_{i,x}$$

$$w\Big|_{\substack{x=0\\ \xi=0}} = w_1 = \cancelto{1}{N_1\big|_0}\, w_1 + \cancelto{0}{N_2\big|_0}\, \theta_1 + \cancelto{0}{N_3\big|_0}\, w_2 + \cancelto{0}{N_4\big|_0}\, \theta_2$$

$$w_{,x}\Big|_{\substack{x=0\\ \xi=0}} = \theta_1 = \cancelto{0}{N_{1,x}\big|_0}\, w_1 + \cancelto{1}{N_{2,x}\big|_0}\, \theta_1 + \cancelto{0}{N_{3,x}\big|_0}\, w_2 + \cancelto{0}{N_{4,x}\big|_0}\, \theta_2$$

$$w\Big|_{\substack{x=L\\ \xi=1}} = w_2 = \cancelto{0}{N_1\big|_L}\, w_1 + \cancelto{0}{N_2\big|_L}\, \theta_1 + \cancelto{1}{N_3\big|_L}\, w_2 + \cancelto{0}{N_4\big|_L}\, \theta_2$$

$$w_{,x}\Big|_{\substack{x=L\\ \xi=1}} = \theta_2 = \cancelto{0}{N_{1,x}\big|_L}\, w_1 + \cancelto{0}{N_{2,x}\big|_L}\, \theta_1 + \cancelto{0}{N_{3,x}\big|_L}\, w_2 + \cancelto{1}{N_{4,x}\big|_L}\, \theta_2$$

4 conditions on N_1 | 4 conditions on N_2 | 4 conditions on N_3 | 4 conditions on N_4

Consider N_1

$$N_1\Big|_{\substack{x=0\\ \xi=0}} = 1 = a_1^{(1)} + a_2^{(1)}(0) + a_3^{(1)}(0) + a_4^{(1)}(0) \Rightarrow a_1^{(1)} = 1$$

$$N_{1,\xi}\Big|_{\substack{x=0\\ \xi=0}} = 0 = a_2^{(1)} + 2a_3^{(1)}(0) + 3a_4^{(1)}(0) = 0 \Rightarrow a_2^{(1)} = 0$$

$$N_1\Big|_{\substack{x=L\\ \xi=1}} = 0 = a_1^{(1)} + a_2^{(1)}(1) + a_3^{(1)}(1) + a_4^{(1)}(1) \Rightarrow 0 = 1 + a_3^{(1)} + a_4^{(1)} \quad \text{—(a)}$$

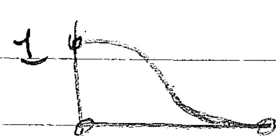
$$N_{1,\xi}\Big|_{\substack{x=L\\ \xi=1}} = 0 = a_2^{(1)} + 2a_3^{(1)}(1) + 3a_4^{(1)}(1) \Rightarrow 0 = 2a_3^{(1)} + 3a_4^{(1)} \quad \text{—(b)}$$

From (b) $\quad a_3^{(1)} = -\tfrac{3}{2}\, a_4^{(1)}$

substitute in (a) $\quad 0 = 1 - \tfrac{3}{2}a_4^{(1)} + a_4^{(1)} \Rightarrow a_4^{(1)} = 2$

$$a_3^{(1)} = -3/2\, a_4 = -3$$

$$N_1 = 1 - 3\xi^2 + 2\xi^3$$



Consider N_2

$$N_2\Big|_{\substack{x=0\\ \xi=0}} = 0 = a_1^{(2)} + a_2^{(2)}(0) + a_3^{(2)}(0) + a_4^{(2)}(0)$$

$$a_1^{(2)} = 0$$

$$N_{2,\xi}\Big|_{\substack{x=0\\ \xi=0}} = 1 = a_2^{(2)} + 2a_3^{(2)}(0) + 3a_4^{(2)}(0)$$

$$a_2^{(2)} = 1$$

$$N_2\Big|_{\substack{x=L\\ \xi=1}} = 0 \quad a_1^{(2)} + a_2^{(2)}(1) + a_3^{(2)}(1) + a_4^{(2)}(1)$$

$$= 0 + 1 + a_3^{(2)} + a_4^{(2)} \quad \text{— (c)}$$

$$N_{2,\xi}\Big|_{\substack{x=L\\ \xi=1}} = 0 = \overset{1}{a_2^{(2)}} + 2a_3^{(2)}(1) + 3a_4^{(2)}(1) \quad \text{— (d)}$$

From (c) $a_3^{(2)} = -1 - a_4^{(2)}$

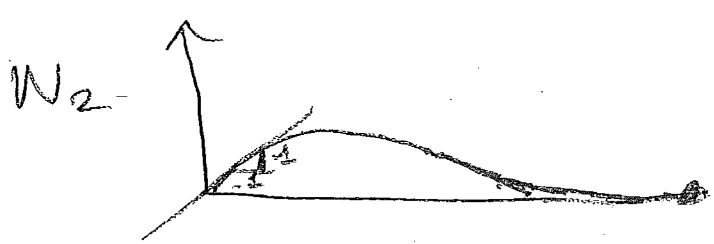
Substitute in (d)

$$1 - 2 - 2a_4^{(2)} + 3a_4^{(2)} = 0$$

$$a_4^{(2)} = 1$$

$$a_3^{(2)} = -1 - 1 = -2$$

$$N_2 = \xi - 2\xi^2 + \xi^3$$



P-Version Shape functions

Reference: F.E. Analysis, B.A. Szabo and I. Babuska, Wiley, 1991

Desire high order shape functions that:

- Have good numerical conditioning (Lagrange polynomials do not work well
- Support the effective construction of element matrices
- Can be used in creating meshes with variable order polynomial order over the domain

In the approach developed they

- selected polynomials with some orthogonality properties that yield better conditioned matrices.
- Support the hierarch construction of elements of increasing order (that is the matrix for the element at order p is a submatrix of the element at order p+1 - our Lagrange shape functions do not do this.

Polynomials that have nice orthogonality properties do not yield interpolating shape functions - makes satisfying the inter element continuity requirement an Tissue.

Ideas we saw with Serendipity elements can help us here.

- We will start with basic C^0 Linear Lagrangian and add things (the way its done here there is no correction stuff needed)

Consider 1-D

$$u^h = a_1 N_1 + a_2 N_2 + \sum_{i=2}^{p} a_{i+1} N_{i+1}$$

We can address the interelement continuity requirement doing the following:

Require:

$$N_1(-1) = 1 \,,\; N_1(1) = 0$$
$$N_2(-1) = 0 \,,\; N_2(1) = 1$$
$$N_i(-1) = 0 \,,\; N_i(1) = 0 \quad \text{for } i = 3(1)\,p+1$$

With this:

$$u^h(-1) = a_1 = u_1$$
$$u^h(1) = a_2 = u_2$$

$N_1 = \frac{1-\xi}{2}$, $N_2 = \frac{1+\xi}{2}$ work for this the question is the remain shape functions which must be higher order – basically up to p and satisfy $N_i(0) = N_i(1) = 0 \quad i = 3(1)p+1$

Shape functions that will do this are constructed from integrals of Legendre Polynomials $P_{i-1}(\xi)$

$$N_{i+1} \equiv \phi_i(\xi) = \sqrt{\frac{2i-1}{2}} \int_{-1}^{\xi} P_{i-1}(t)\,dt \qquad i = 2(1)p$$

where the Legendre polynomials are:

$$
\begin{aligned}
P_0(t) &= 1 \\
P_1(t) &= t \\
P_2(t) &= \tfrac{1}{2}(3t^2 - 1) \\
P_3(t) &= \tfrac{1}{2}(5t^3 - 3t) \\
P_4(t) &= \tfrac{1}{8}(35t^4 - 30t^2 + 3) \\
P_5(t) &= \tfrac{1}{8}(63t^5 - 70t^3 + 15t) \\
&\vdots \\
P_n(t) &= \frac{2n-1}{n}\, t\, P_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t)
\end{aligned}
$$

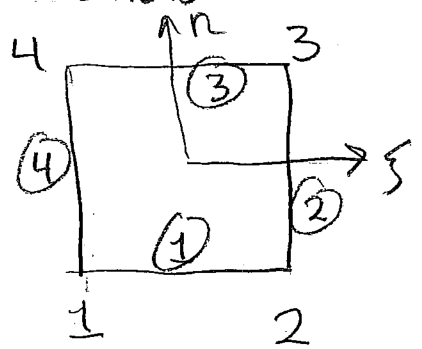
Not obvious – but you can check $\phi(-1) = \phi(1) = 0$

The $\sqrt{\frac{2i-1}{2}}$ is a normalization such that

$$\int_{-1}^{1} \phi_{i,\xi}\, \phi_{j,\xi}\, d\xi = \delta_{ij} \quad \left(\text{note the orthogonality}\right)$$

The basic ideas can be extended to 2-D and 3-D

Consider a quadrilaterial elements



For the base linear element use our standard linear shape functions

$$N_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

Then much like Serendipity elements we will add modes to the sides.

Side modes: The are $4(p-1)$ side mode shape functions for $p \geq 2$

For side 1:

$$N_i^{(1)} = \frac{1}{2}(1-\eta)\,\phi_i(\xi) \qquad i = 2(1)p$$

For side 2:

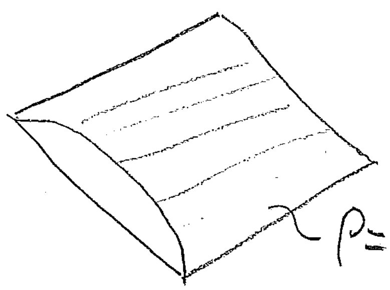
$$N_i^{(2)} = \frac{1}{2}(1+\xi)\,\phi_i(\eta) \qquad i = 2(1)p$$

For side 3:

$$N_i^{③} = \frac{1}{2}(1+\eta)\,\phi_i(\xi) \qquad i = 2(1)p$$

For side 4:

$$N_i^{④} = \frac{1}{2}(1-\xi)\,\phi_i(\eta) \qquad i = 2(1)p$$



Note that the dof a_i's associated with the side modes are not values of u at specific points - They are simple dof that are shared (with corrections for signs for odd-order terms) between elements - This gives C^0 continuity with no corrections needed.

Recall that with doing these side modes like this (linear in the other direction) we never get:

$$\xi^2\eta^2,\quad \xi^3\eta^2,\quad \xi^2\eta^3,\quad \xi^4\eta^2 \ldots \text{ etc.}$$

to get these we also need face modes

Face modes

There are $\frac{1}{2}(p-2)(p-3)$ face modes for $p \geq 4$

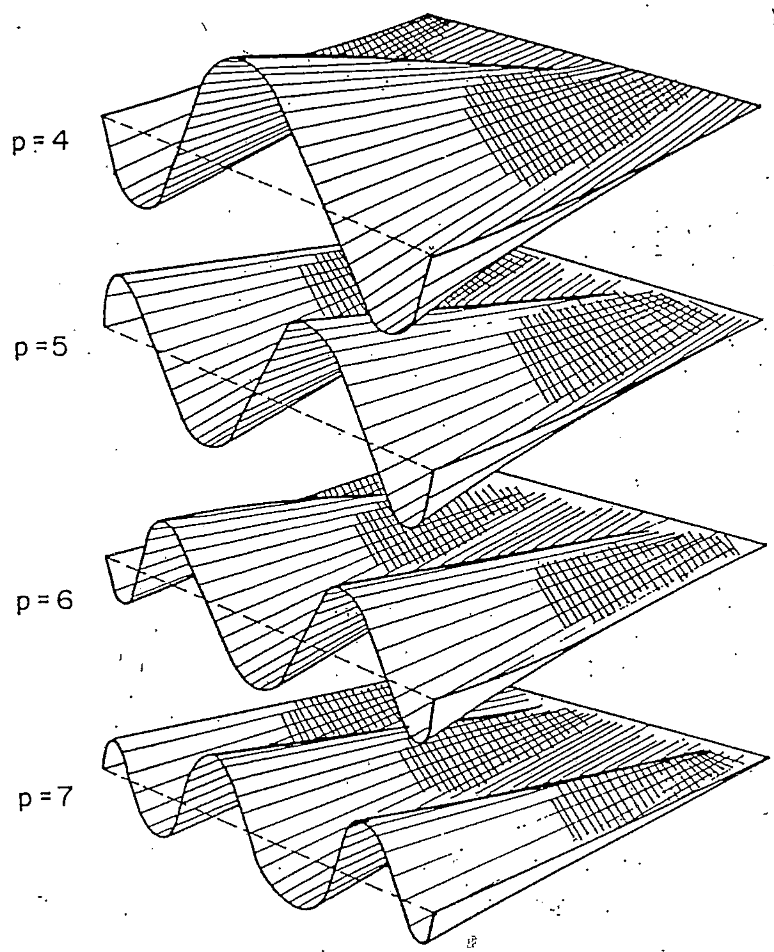
The first one is:

$$N_1^{(0)}(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

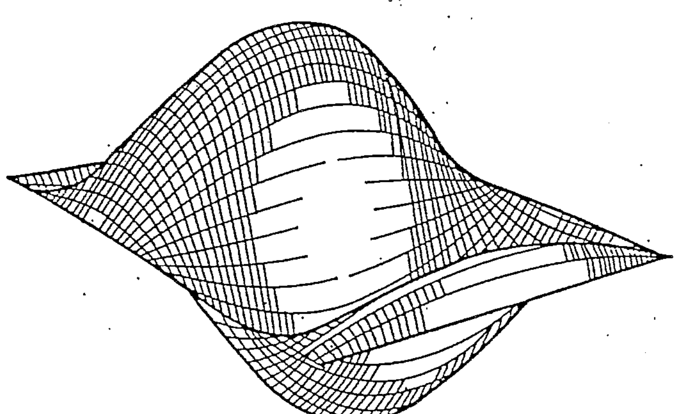
From there the higher modes are obtained by multiplying by Legendre polynomials

$$N_2^{(0)} = N_1^{(0)}(\xi,\eta)\; P_1(\xi)$$
$$N_3^{(0)} = N_1^{(0)}(\xi,\eta)\; P_1(\eta)$$
$$N_4^{(0)} = N_1^{(0)}(\xi,\eta)\; P_2(\xi)$$
$$N_5^{(0)} = N_1^{(0)}(\xi,\eta)\; P_1(\xi)\, P_1(\eta)$$
$$\vdots$$

P7



Edge modes



Face mode

Isogeometric Analysis – motivation & definition

- reduce effort of geometry conversion from CAD into a suitable mesh for FEA

- ISOPARAMETRIC (FE-Analysis)
  use same approximation for geometry and deformation
  (normally: low order Lagrange polynomials ---- in LS-DYNA basically only linear elements)

GEOMETRY	$\leftrightarrow$	DEFORMATION

- ISOGEOMETRIC (CAD - FEA)
  same description of the geometry in the design (CAD) and the analysis (FEA)

CAD	$\leftrightarrow$	FEA

- common geometry descriptions in CAD
  - NURBS (Non-Uniform Rational B-splines) → most commonly used
  - T-splines → enhancement of NURBS
  - subdivision surfaces → mainly used in animation industry
  - and others

Reality is FEM will not Drive CAD — CAD has much more to do than make meshes — meshing must work with CAD

Isogeometric Analysis: Toward Integration of CAD and FEA
J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs, Wiley 2009

From B-splines to NURBS

- B-spline basis functions
  - constructed recursively
  - positive everywhere (in contrast to Lagrange polynomials)
  - shape of basis functions depend on: knot-vector and polynomial degree
  - knot-vector: non-decreasing set of coordinates in parameter space
  - normally $C^{(P-1)}$-continuity
    - → e.g. lin. / quad. / cub. / quart. Lagrange: → C^0 / C^0 / C^0 / C^0
    - → e.g. lin. / quad. / cub. / quart. B-spline: → C^0 / C^1 / C^2 / C^3

Note: Partition of unity Condition met –

$$\sum_{i=1}^{n} N_{i,p} = 1$$

Example of a uniform knot-vector: $\Xi = \{0,1,2,3,4,\dots\}$

$p = 0:$

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Cox - de Boor Recursion formula

$p > 0:$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1(\xi)} + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1(\xi)}$$

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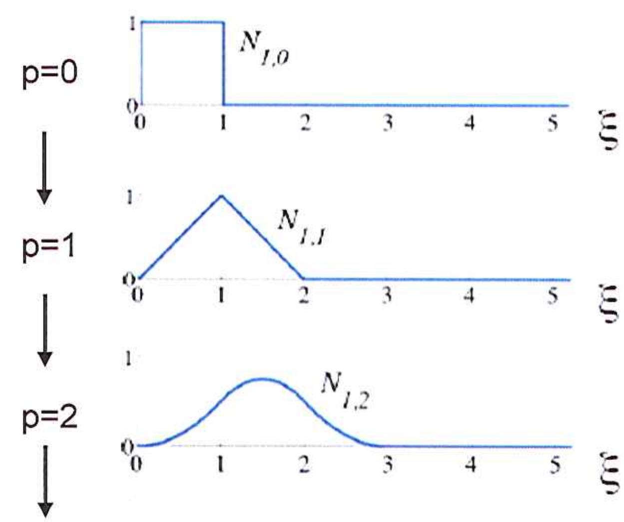


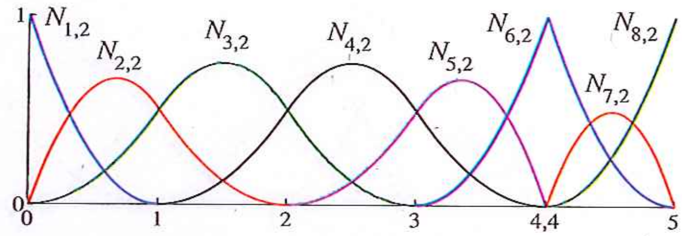


Figure 2.5 Quadratic basis functions for open, non-uniform knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5\}$.

The use of a non-uniform knot vector allows us to obtain much richer behavior than is possible with a simple uniform one. A quadratic example is presented in Figure 2.5 for the open, non-uniform knot vector $\Xi = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10}, \xi_{11}\} = \{0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5\}$. Note that the basis functions are interpolatory at the ends of the interval and also at $\xi = 4$, the location of a repeated knot. At this repeated knot, only C^0-continuity is attained. Elsewhere, the functions are C^1-continuous. In general, basis functions of order p have $p - m_i$ continuous derivatives across knot ξ_i, where m_i is the multiplicity of the value of ξ_i in the knot vector. When the multiplicity of a knot value is exactly p, the basis is interpolatory at that knot. When the multiplicity is $p + 1$, the basis becomes discontinuous and the patch boundary is formed.

This relationship between continuity and the multiplicity of the knots is even more apparent in Figure 2.6, in which we have a fourth order curve with differing levels of continuity at every element boundary. At the first internal element boundary, $\xi = 1$, the knot value appears only once in the knot vector, and so we have the maximum level of continuity possible: $C^{p-1} = C^3$. At each subsequent internal knot value, the multiplicity is increased by one, and so the number of continuous derivatives is decreased by one. Note, as before, that when a knot

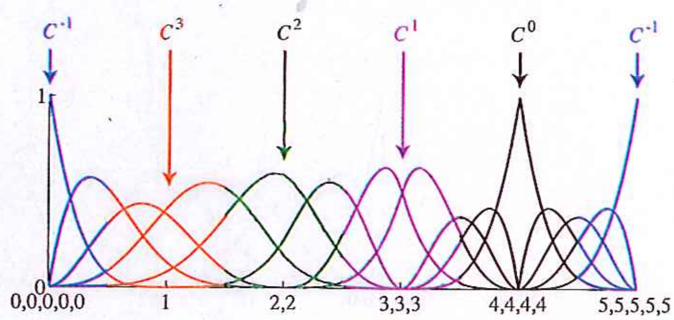


Figure 2.6 Quartic ($p = 4$) basis functions for an open, non-uniform knot vector $\Xi = \{0, 0, 0, 0, 0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5\}$. The continuity across an interior element boundary is a direct result of the polynomial order and the multiplicity of the corresponding knot value.

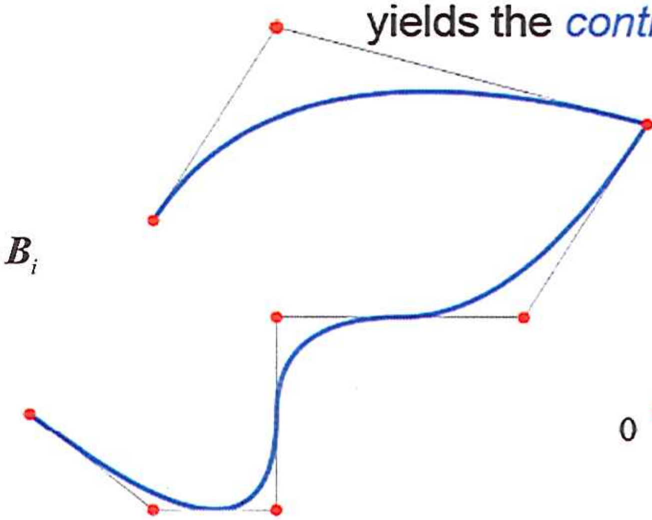
From B-splines to NURBS

- B-spline curves
  - control points $\boldsymbol{B}_i$ / control polygon (control net)
  - knots

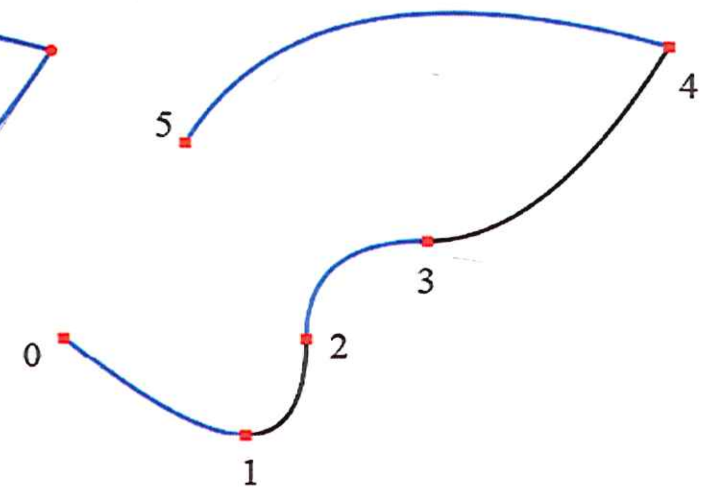
Linear interpolation of control points yields the *control polygon*

linear combination:

$$C(\xi)=\sum_{i=1}^{n} N_{i,p}(\xi)\boldsymbol{B}_i$$

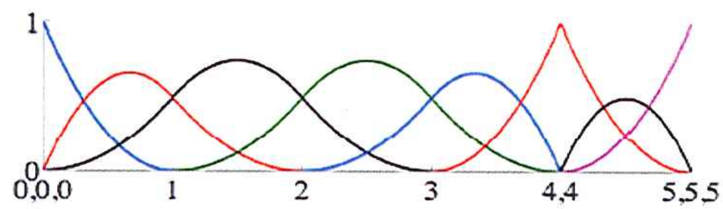


● - control points



■ - knots

Quadratic basis



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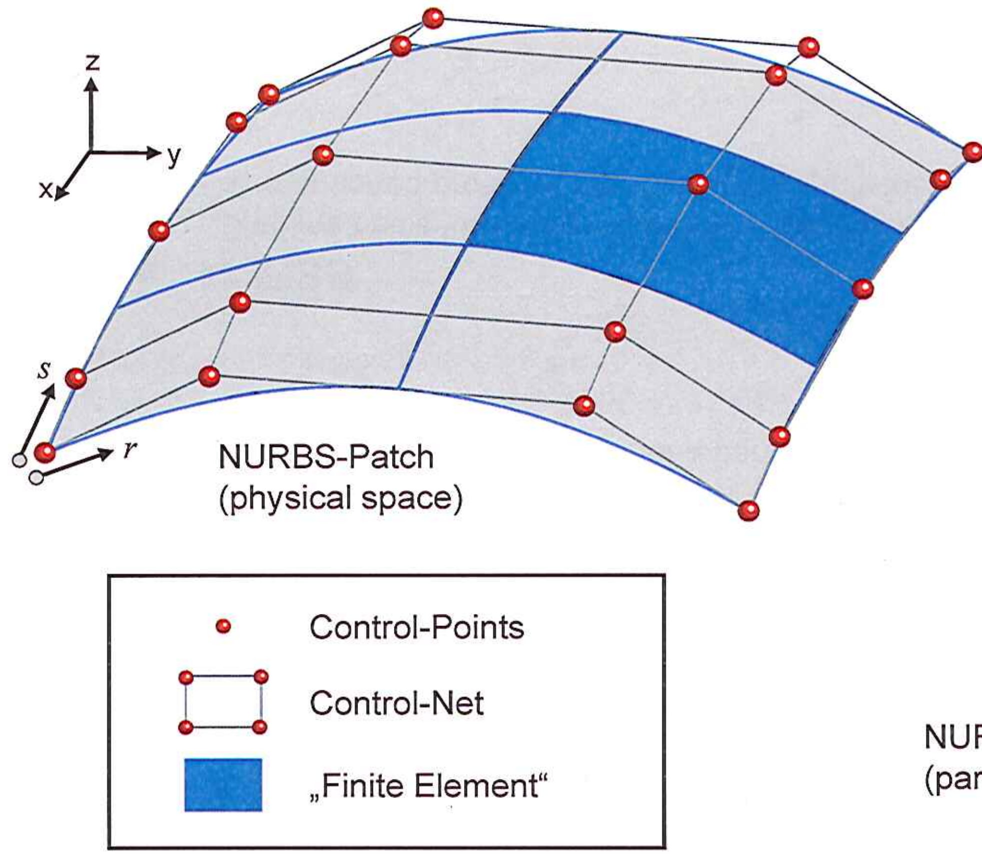
NURBS-based finite elements in LS-DYNA

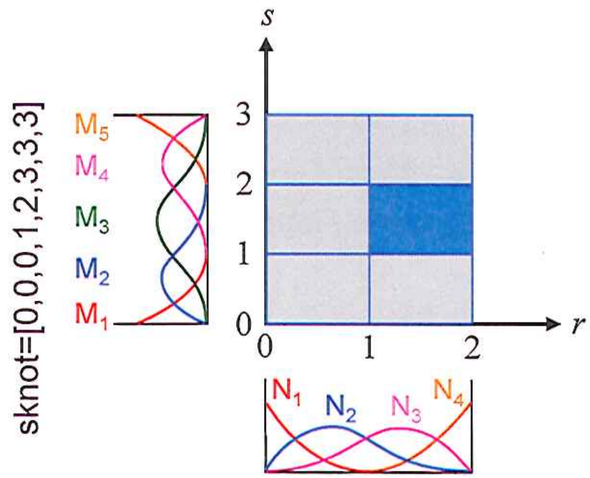
- A typical NURBS-Patch and the definition of elements
  - elements are defined through the knot-vectors (interval between different values)
  - shape functions for each control-point

$$S(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)\, M_{j,q}(\eta)\, B_{ij}$$

Again – a partition of unity



polynomial order:
- quadratic in r-direction (pr=2)
- quadratic in s-direction (ps=2)



NURBS-Patch
(parameter space)

From B-splines to NURBS - summary

- B-spline basis functions
  - recursive
  - dependent on knot-vector and polynomial order
  - normally $C^{(P-1)}$-continuity
  - „partition of unity“ (like Lagrange polynomials)
  - refinement (h/p and k) without changing the initial geometry → adaptivity
  - control points are normally not a part of the physical geometry (non-interpolatory basis functions)

- NURBS
  - B-spline basis functions + control net with weights
  - all mentioned properties for B-splines apply for NURBS

knot insertion – like h-refinement
Order elevation – like p-refinement
k-refinement – increase order of continuity between elements

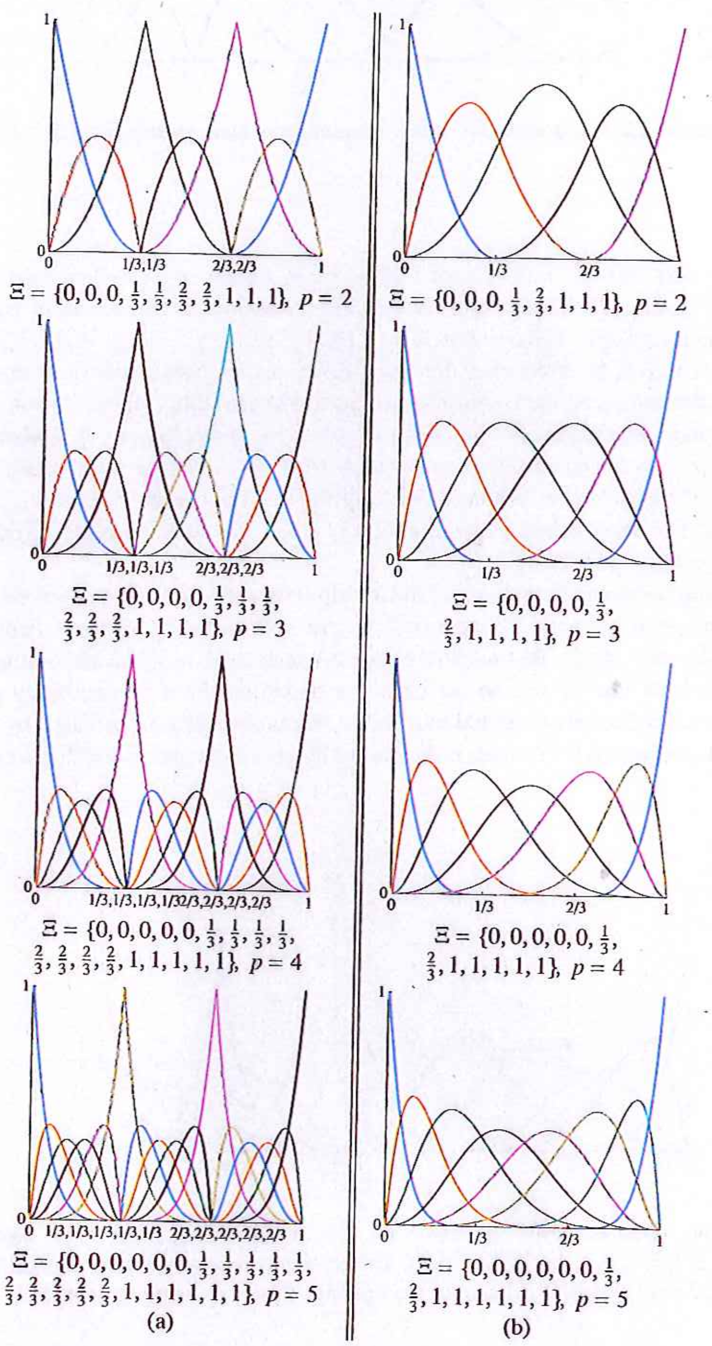


Figure 2.24 Three element, higher-order meshes for p- and k-refinement. (a) The p-refinement approach results in many functions that are C^0 across element boundaries. (b) In comparison, k-refinement results in a much smaller number of functions, each of which is C^{p-1} across element boundaries.

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2.1.4.4 The *hpk*-r

As we have shown
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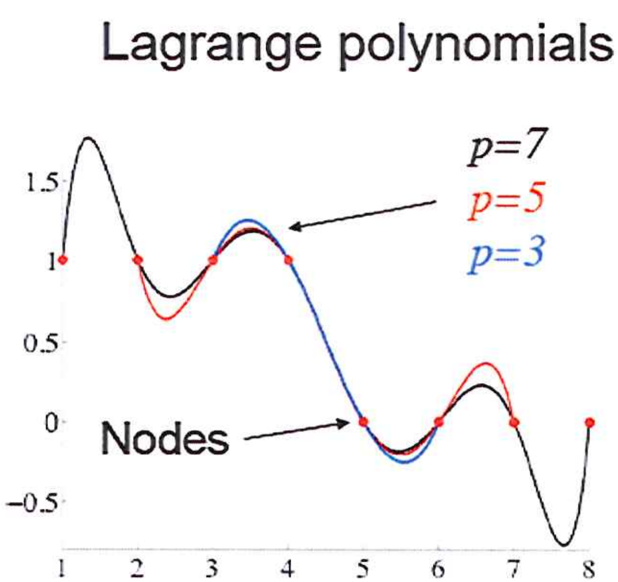
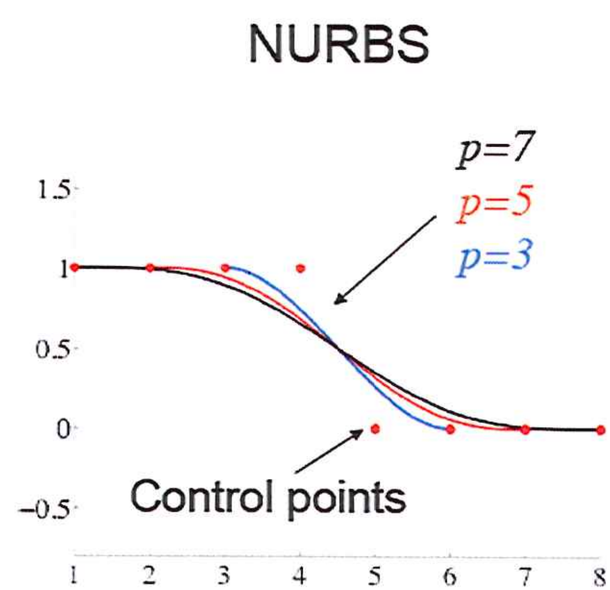
From B-splines to NURBS

- Smoothness of Lagrange polynomials vs. NURBS





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(a) (b)

Figure 2.14 Interpolation with Lagrange polynomials. (a) The points to be interpolated are accurate to six digits after the decimal point. (b) The points to be interpolated are accurate to only four digits after the decimal point.

where $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$ are univariate B-spline basis functions of order p and q, corresponding to knot vectors Ξ and $\mathcal{H}$, respectively.

Many of the properties of a B-spline surface are the result of its tensor product nature. The basis is pointwise nonnegative, and forms a partition of unity as $\forall(\xi, \eta) \in [\xi_1, \xi_{n+p+1}] \times [\eta_1, \eta_{m+q+1}]$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta) = \left(\sum_{i=1}^{n} N_{i,p}(\xi)\right)\left(\sum_{j=1}^{m} M_{j,q}(\eta)\right) = 1. \tag{2.18}$$

The number of continuous partial derivatives in a given parametric direction may be determined from the associated one-dimensional knot vector and polynomial order. The surface again possesses the property of affine covariance and has a strong convex hull property. Interestingly, there is no known variation diminishing property for surfaces, though the convex hull property precludes any two-dimensional analogues of the types of oscillations we saw in Figure 2.13a, thus generalizing the result of Figure 2.13b to multiple dimensions.

The local support of the basis functions also follows directly from the one-dimensional functions that form them. The support of a given bivariate function $\tilde{N}_{i,j;p,q}(\xi, \eta) = N_{i,p}(\xi)M_{j,q}(\eta)$ is exactly $[\xi_i, \xi_{i+p+1}] \times [\eta_j, \eta_{j+q+1}]$. Let us consider a specific example of a biquadratic ($p = q = 2$) surface formed from knot vectors $\Xi = \{0, 0, 0, 0.5, 1, 1, 1\}$ and $\mathcal{H} = \{0, 0, 0, 1, 1, 1\}$, with control points listed in Table 2.1, resulting in the control net and mesh shown in Figure 2.15. For this case, the support of $\tilde{N}_{1,1;2,2}(\xi, \eta)$, is $[\xi_1, \xi_4] \times [\eta_1, \eta_4]$. Similarly, the support of $\tilde{N}_{3,2;2,2}(\xi, \eta)$, for example, is $[\xi_3, \xi_6] \times [\eta_2, \eta_5]$. The support of each of these functions is shown in the index space in Figure 2.16a. By equally spacing each of the knots in the plot, it is easy to see exactly which knot spans each of the functions are supported in, including where they overlap. Such a viewpoint is very useful when developing algorithms (see Appendix A at the end of the book for a discussion of the index space and so-called "NURBS coordinate" in the context of connectivity). Alternatively, we can present the same information in the parameter space, as in Figure 2.16b. Here, we have taken into account the actual knot values. It is clear that we only have two nontrivial elements (elements with positive measure), and therefore only two elements in which calculations need to be performed during analysis. Function $\tilde{N}_{3,2;2,2}(\xi, \eta)$ has support in both of these elements, while $\tilde{N}_{1,1;2,2}(\xi, \eta)$ is only

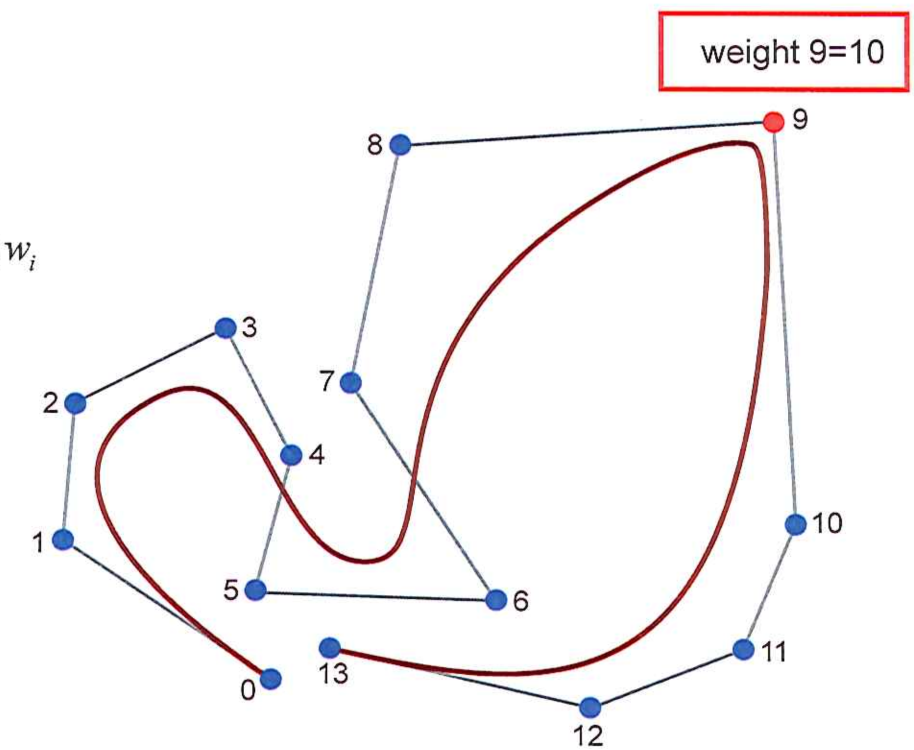
From B-splines to NURBS

- NURBS – Non-Uniform Rational B-splines
  - weights at control points leads to more control over the shape of a curve
  - projective transformation of a B-spline

$$R_i^p(\xi) = \frac{N_{i,p}(\xi)\, w_i}{W(\xi)}$$

$$with:\ W(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\, w_i$$

$$C(\xi) = \sum_{i=1}^{n} R_i^p(\xi)\, \boldsymbol{B}_i$$

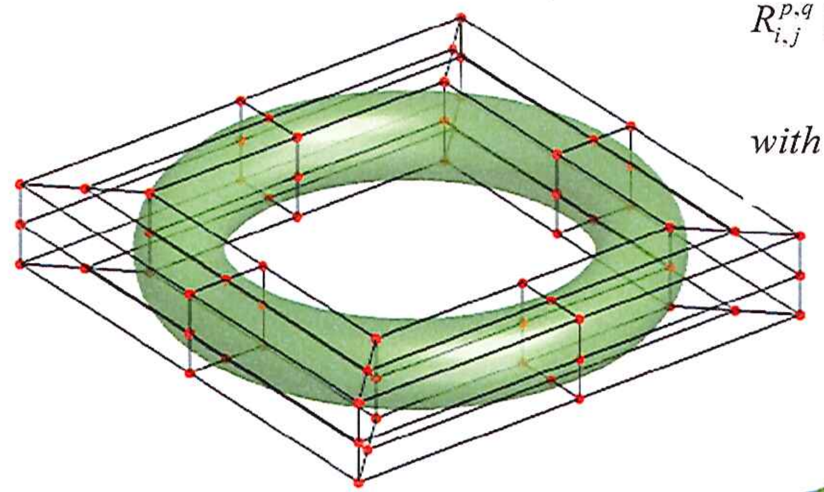


From B-splines to NURBS

- NURBS – surfaces (tensor-product of univariate basis)

$$R_{i,j}^{p,q}(\xi,\eta)=\frac{N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}{W(\xi,\eta)}$$

$$with: W(\xi,\eta)=\sum_{i=1}^{n}\sum_{j=1}^{m}N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}$$



Control net

Mesh

T.J.R. Hughes

after all transparencies except last one

By using the same shape functions for

$\underset{\sim}{x} = C(\xi)$ in 2-D for curves
$\underset{\sim}{x} = \underset{\sim}{S}(\xi)$ in 3-D for surfaces

we have the same analysis for condition C3 of exactly representing a linear function - that is satisfaction of the partition ~~of~~ unity $\Sigma N = 1$

When used as finite element basis you have a bit different definition of what things mean in terms of the coefficients in the expansion

$$\hat{u}^h(\xi) = \sum_{A=1}^{n_{np}} N_A(\xi)\, d_A$$

u^h in (ξ) space

d_A - control variables (not nodal values)

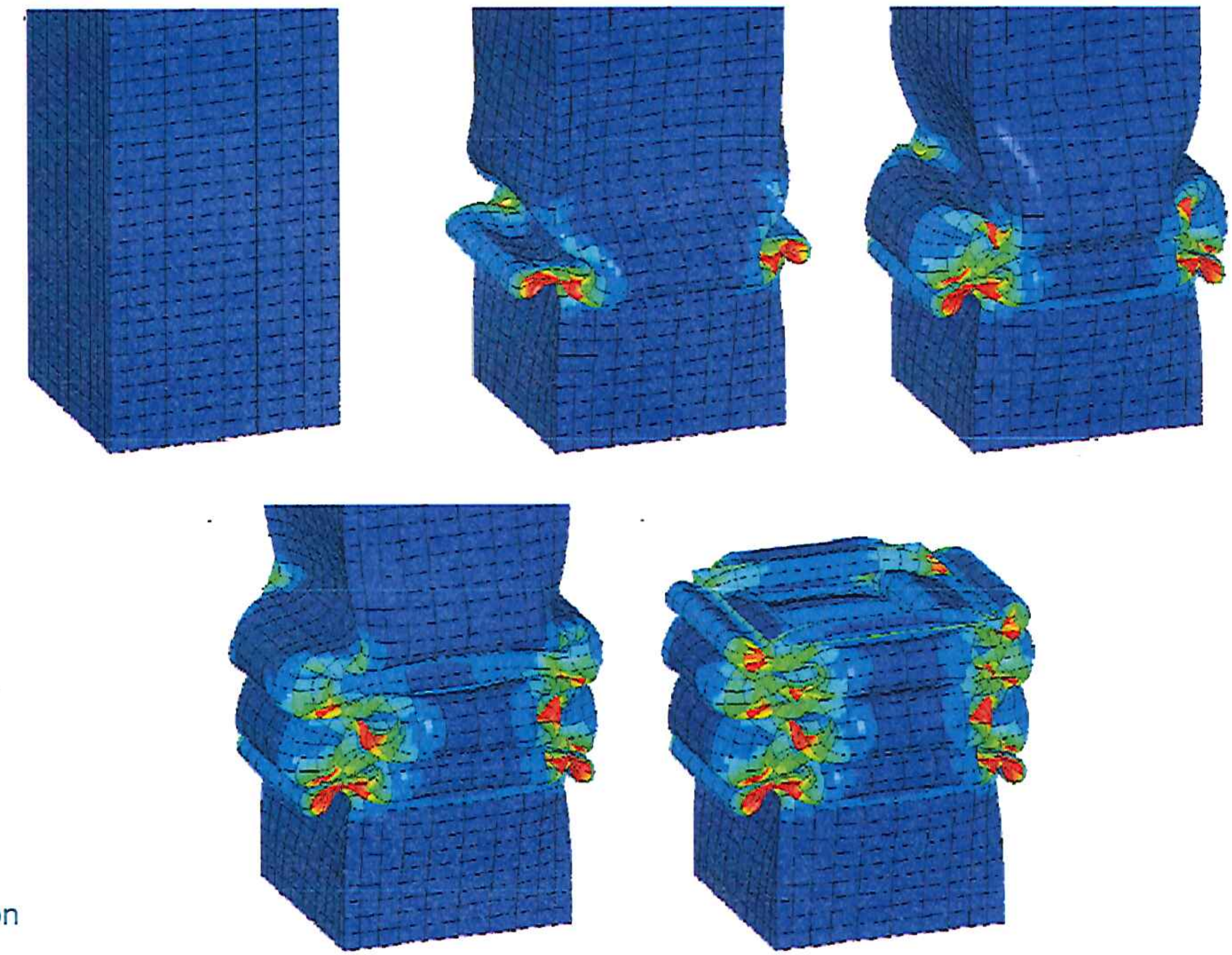
however we can use the inverse mapping to get what we want

$$u^h = \hat{u}^h \circ \underset{\sim}{x}^{-1}$$

in real space

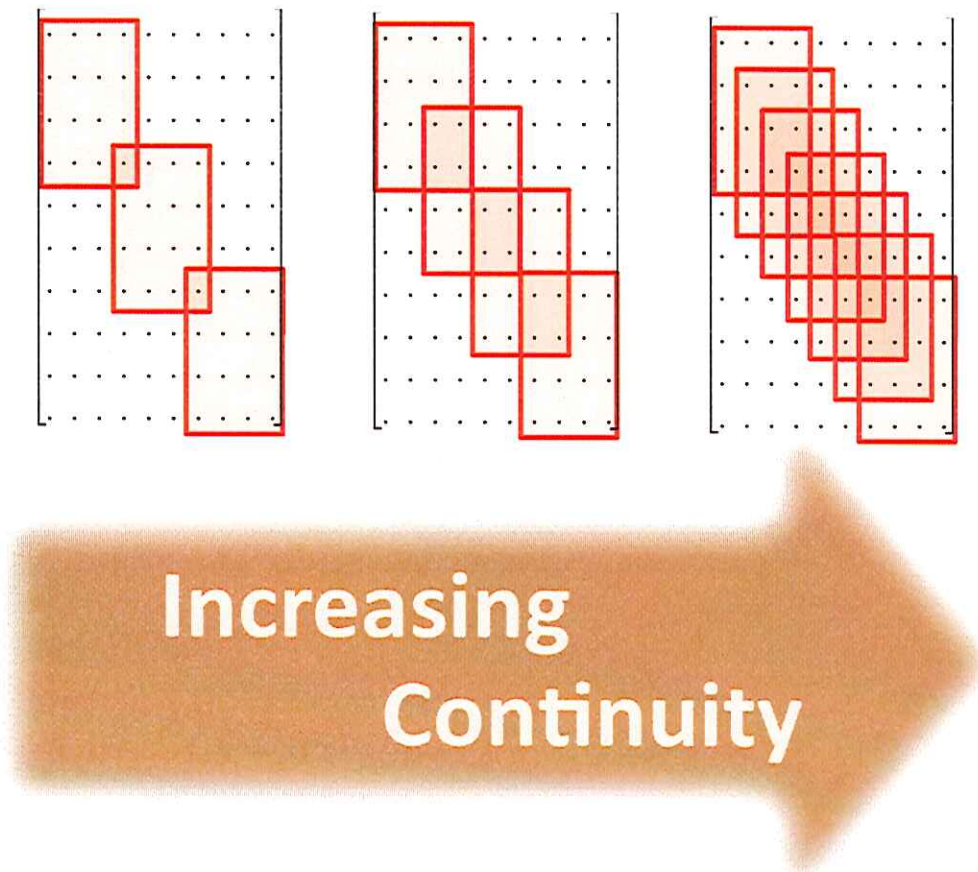
This does lead to needing to do extra work to deal with nonzero essential BC.

Square Tube Buckling



D.J. Benson

The high price of continuity



Higher continuous basis result in element stiffness matrix blocks overlapping, causes performance loss of multi-frontal algorithm