Accuracy of the Finite Element method

We will cover Appendix 4.I.2-4.I.4 (already covered 4.I.1), section 4.1 and some added material.

We will not attempt to go into full mathematical detail, our more modest goals are:

- Introduce you to a bit more of the terminology you will see when you read papers
- Inform you of the basics convergence properties of finite element methods.

Appendix 4.I.2 Sobolev Norms

Consider a domain $\Omega \subset \Re^{n_{sd}}$, $n_{sd} \geq 1$ (will be the spatial dimension – 1D, 2D, 3D), and let $u, v : \Omega \rightarrow \Re$ (note – scalar fields)

The $L_2(\Omega)$ (or equivalently $H^o(\Omega)$) inner product and norm are defined by

$$(u,v) = (u,v)_0 = \int_\Omega uv \, d\Omega$$

$$\|u\| = (u,u)^{1/2}$$

The $H^1(\Omega)$ inner product and norm are defined by

$$(u,v)_1 = \int_\Omega (uv + u_{,i} v_{,i}) \, d\Omega \text{ (sum } 1 \leq i \leq n_{sd} \text{)}$$

$$\|u\|_1 = (u,u)_1^{1/2}$$

The $H^s(\Omega)$ inner product and norm are defined by

$$(u,v)_s = \int_\Omega \left(uv + u_{,i} v_{,i} + u_{,ij} v_{,ij} + \ldots + u_{,ij\ldots k} v_{,ij\ldots k} \right) d\Omega$$

where the $u_{,ij\ldots k} v_{,ij\ldots k}$ indicates taking s derivatives

$$\|u\|_s = (u,u)_s^{1/2}$$

When dealing with vector fields $\vec{u}$ and $\vec{v}$ we have to account for components so in that case:
The $H^s(\Omega)$ inner product and norm are defined by

$$(\vec{u},\vec{v})_s = \int_\Omega \left(u_i v_i + u_{i,j} v_{i,j} + u_{i,jk} v_{i,jk} + \ldots + u_{i,jk\ldots l} v_{i,jk\ldots l} \right) d\Omega$$

where the $u_{i,jk\ldots l} v_{i,jk\ldots l}$ indicates taking s derivatives on the components

$$\|\vec{u}\|_s = (\vec{u},\vec{u})_s^{1/2}$$

There are time one refers to the semi norm, which is just the last term

$$|\vec{u}|_s = \left(\int_\Omega u_{i,jk\ldots l} u_{i,jk\ldots l} \, d\Omega \right)^{1/2}$$

The s^{th} Sobolev space, denoted H^s, is the collection of functions $\vec{u} : \Omega \rightarrow \Re^n$ where n is the number of components of $\vec{u}$ which is $\geq n_{sd}$, it is $= n_{sd}$ in things we have been looking at. You will also hear the term Hilbert spaces in that Sobolev spaces are Hilbert spaces.

Note that Hilbert spaces satisfy the inclusion property:

$$H^r \subset H^s, r > s$$

For example if you can take up through 6 derivatives, you can certainly take 4 or 5.

Sobolev Imbedding Theorems

A key result of importance to us is to relate these integral space properties to the more classical definition of continuous functions.

Definition (of a continuous function): Let $C_b^k, k \geq 0$, be the space of functions $\vec{u} : \Omega \rightarrow \Re^n$ which are:

- Bounded (that is less than infinity)
- Continuous, and have continuous and bounded derivatives of order j, $1 \leq j \leq k$

Sobolev Imbedding Theorem: If Ω is an open set in $\Re^{n_{sd}}$ (and not a hyper surface – that is "full dimension" and not something like just a surface in 3D) and $s > n_{sd}/2 + k$ (note the dependence on n_{sd}) then

$$H^s \subset C_b^k$$

Examples:

$\Omega = \left]a, b\right[\subset \Re^1$. Then for H^1 we have $1 > 1/2 + k$ thus $k = 0$ and we are working with C_b^0 functions, for H^2 we have $2 > 1/2 + k$ thus $k = 1$ and we are working with C_b^1 functions.

$\Omega = \left]bi-unit-sqaure\right[\subset \Re^2$. Then for H^2 we have $2 > \frac{2}{2} + k$ thus $k = 0$ and we are working with C_b^0 functions. Note with the increased dimension we only get C_b^0 and not C_b^1. The dimension of the domain makes a difference (or more simply, 1D is easier than 2D and 3D).

The imbedding theorem is sharp in the sense that if $s \le \frac{n_{sd}}{2} + k$, there exists H^s functions that are not C_b^k.

The range of space, $\Re^n$ (n is the number of components of $\vec{u}$ which is $\ge n_{sd}$ in the vector case) plays no role in the imbedding theorem.

4.I.3 Approximation properties of finite element spaces in Sobolev norms

We want to answer the question: Given a Sobolev space H^r and a finite element space δ^h, how well are we able to approximate $u \subset H^r$ if we are allowed to pick any member of δ^h? To go through this in detail is beyond what we are prepared to cover, thus we will state the key results and indicate their importance.

The key result we want to state will indicate how the error decreases (in terms of convergence rate) as we "refine" the mesh, where, for now, refine means to make elements of a given order finite element mesh smaller.

Consider a set of finite elements in the finite dimensional space δ^h where we have some measure of the element size, h, as $\left\{\delta^h \mid h \subset]0,D]\right\}$ where D is the diameter of the element (more on this later). If we have two meshes with element sized h^1 and h^2, where $h^1 < h^2$, we can think of δ^{h_1} as a refinement of δ^{h_2}, with the dimension of δ^{h_1} being greater than δ^{h_2}.

A common example of this is you have an initial mesh δ^{h_2} and you create the refined mesh by subdividing the elements in half (in each dimension) to create the mesh δ^{h_1} which has elements "½ the size".

A definition and the first key result (just stated):
A collection of finite element spaces is k,m-regular (or simply regular) if for each fixed h (note - k will be the order on complete polynomial for our finite element and m is to highest order partial in our energy functional)

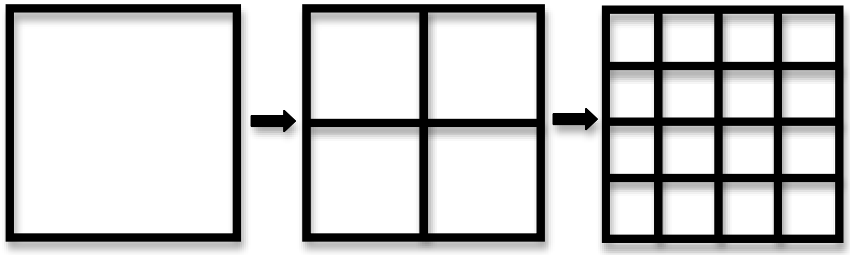
a) $\delta^h \subset H^m$
b) For every $u \subset H^r$, $r \geq 0$, and for all s such that $0 \leq s \leq \min\{r,m\}$, there exists an approximate solution, $U^h \in \delta^h$ and constant c, independent of u and h, such that

$$\left\|u - U^h\right\|_s \leq ch^\alpha \left\|u\right\|_r \quad \text{eq.1}$$

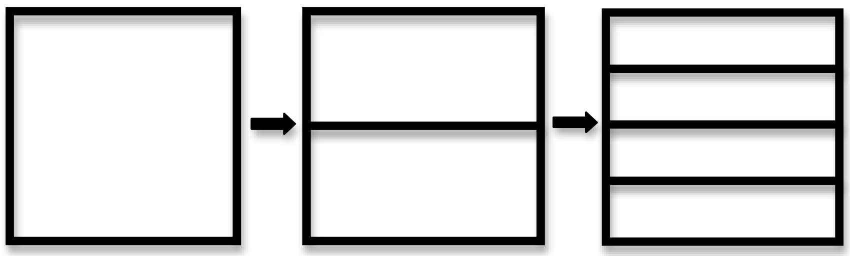
where $\alpha = \min(k+1-s, r-s)$ is the convergence rate

Remarks:

- This result is a cornerstone of the FE error equation.
- The essence of eq. 1 is that the more refined the mesh, the better the solution.
- The highest rate of convergence is in the H^0 norm of u while the rate of convergence decrease with increasing s. This means that the first derivatives converge at a rate one less than the value (in the norm) and the second derivatives one less than the first derivatives, etc.
- The rate of convergence is influence by the “smoothness” of the exact solution, u. If u is smooth enough, the rate is dictated by the order of the finite element, k. If it is not smooth enough it is dictated by the smoothness of the exact solution, r. This is of great importance to us since in many problems of interest we have analytic singularities (singularities with finite energy so we can solve them) where $r < k + 1$.
- Convergence of the solution only guaranteed if the refinement is quasi-uniform which maintains a limit on the minimum element aspect ratio – elements must get smaller in all directions.



Uniform refinement



Not quasi-uniform refinement

Examples:

Let δ^h be a piecewise linear FE space ($k=1$) over N elements. The collection $\{\delta^h \mid h=1/N\}$, where N is a positive integer, is 1,1-regular. That is for $h=1/N$

- $\delta^h \subset H^1$
- For every $u \subset H^2$ (making sure u is smooth enough) there exists a $U^h \in \delta^h$ such that

$$\left\|u-U^h\right\|_0 \le ch^2\left|u\right|_2$$

$$\left\|u-U^h\right\|_1 \le ch\left|u\right|_2$$

where it happens to be a semi norm on the right which is even tighter than the full norm.

Let δ^h be a piecewise quadratic FE space ($k=2$) over N elements. The collection $\{\delta^h \mid h=1/N\}$, where N is a positive integer, is 2,1-regular. That is for $h=1/N$

- $\delta^h \subset H^1$
- For every $u \subset H^3$ there exists a $U^h \in \delta^h$ such that

$$\left\|u-U^h\right\|_0 \leq ch^3\left\|u\right\|_3$$

$$\left\|u-U^h\right\|_1 \leq ch^2\left\|u\right\|_3$$

Let δ^h be a Hermite cubic FE space ($k=3$ with continuous value and first derivative) over N elements to be used for the beam problem. The collection $\{\delta^h \mid h=1/N\}$ where N is a positive integer is 3,2-regular. That is for $h=1/N$

- $\delta^h \subset H^2$
- For every $u \subset H^4$ there exists a $U^h \in \delta^h$ such that

$$\left\|u-U^h\right\|_0 \leq ch^4\left|u\right|_4$$

$$\left\|u-U^h\right\|_1 \leq ch^3\left|u\right|_4$$

$$\left\|u-U^h\right\|_2 \leq ch^2\left|u\right|_4$$

Let δ^h be an isoparametric FE space of complete order k on a domain $\Omega \subset \Re^{n_{sd}}, n_{sd} \geq 1$. In the case where there is no geometric approximation of the mesh to the domain ($\bar{\Omega} \equiv \bigcup_{e=1}^{n_{el}} \bar{\Omega}^e$) and only slightly curved elements for $k \geq 2$, the finite element space, δ^h, is k,1-regular

- $\delta^h \subset H^1$
- For every $u \subset H^{k+1}$ there exists a $U^h \in \delta^h$ such that

$$\left\|u-U^h\right\|_0 \leq ch^{k+1}\left\|u\right\|_{k+1}, \quad \left\|u-U^h\right\|_1 \leq ch^k\left\|u\right\|_{k+1}$$

4.I.4 Hypotheses on $a(\bullet,\bullet)$ - use of equivalence of norms

To make use of the stuff we have seen so far we need to relate it to our finite element problem

$$a(\vec{w},\vec{u}) = (\vec{w},\vec{f}) + (\vec{w},\vec{h})_\Gamma \quad \forall \vec{w} \in V \subset H_0^m$$

The subscript 0 on H_0^m is to indicate the satisfaction of the homogeneous (zero) version of the essential B.C.

Definition: Two norms, $\|\bullet\|^{(1)}$ and $\|\bullet\|^{(2)}$, on a linear space A, are referred to as equivalent if there exists constants c_1 and c_2 such that $\forall x \in A$

$$c_1 \|x\|^{(1)} \le \|x\|^{(2)} \le c_2 \|x\|^{(1)}$$

Our hypothesis is that we will assume $\|\vec{w}\|_m$ and $a(\vec{w},\vec{w})^{1/2}$ define equivalent norms $\forall \vec{w} \in V \subset H_0^m$

$$c_1 \|\vec{w}\|_m \le a(\vec{w},\vec{w})^{1/2} \le c_2 \|\vec{w}\|_m$$

This will be true for all the $a(\vec{w},\vec{w})$ we have seen so long as the associated material tensors (e.g., $\kappa, EI, \kappa_{ij}, c_{ijkl}$) are positive definite.

A quick summary of what we have form this:

- Under the right conditions, refining the mesh increase the accuracy and we converge to the exact solution in the limit ($h \to 0$)
- The rate of convergence is a function of the highest order complete polynomial order, or the space of the exact solution (whichever is more restrictive).
- For smooth exact solutions our convergence rate increase as we increase the elements complete polynomial order.
- The higher the order of derivative desired the lower the rate of convergence.

Section 4.1 Standard FE Error Estimate

Our weak form: Find $u \in \delta$ such that

$$a(\vec{w},\vec{u}) = (\vec{w},\vec{f}) + (\vec{w},\vec{h})_\Gamma \quad \forall \vec{w} \in V \subset H_0^m$$

Our finite element Galerkin form: Find $u^h \in \delta^h$ such that

$$a(\vec{w}^h,\vec{u}^h) = (\vec{w}^h,\vec{f}) + (\vec{w}^h,\vec{h})_\Gamma \quad \forall \vec{w}^h \in V^h \subset H_0^m$$

where

- $\delta^h \subset \delta$, $V^h \subset V$
- $a(\bullet,\bullet),(\bullet,\bullet),(\bullet,\bullet)_\Gamma$ are symmetric and bilinear (will use those properties in proofs)
- $a(w,w) \geq 0$ and $a(w,w) = 0$ iff $w = 0$ (pos. definite)
- $\|\vec{w}\|_m$ and $a(\vec{w},\vec{w})^{1/2}$ define equivalent norms on V

Define the error as $\vec{e} = \vec{u}^h - \vec{u}$. We will show that

(A) $a(\vec{w}^h,\vec{e}) = 0 \quad \forall \vec{w}^h \in V^h$. The error is orthogonal to our weighting space WRT $a(\bullet,\bullet)$. This says that $\vec{u}^h$ is the projection of $\vec{u}$ onto V^h WRT $a(\bullet,\bullet)$.

(B) $a(\vec{e},\vec{e}) \leq a(\vec{U}^h - \vec{u},\vec{U}^h - \vec{u}), \forall \vec{U}^h \in \delta^h$. This is the best approximation property – says the FE solution is the best, WRT $a(\bullet,\bullet)$,since no other member of δ^h give a smaller error norm value.

(C) In the case of homogeneous essential boundary conditions (where we have $\delta^h = V^h$) we will have $a(\vec{u},\vec{u}) = a(\vec{u}^h,\vec{u}^h) + a(\vec{e},\vec{e})$ and $a(\vec{u}^h,\vec{u}^h) \leq a(\vec{u},\vec{u})$

Proof of (A)
From the FE solution we have:

$$a(\vec{w}^h,\vec{u}^h) = (\vec{w}^h,\vec{f}) + (\vec{w}^h,\vec{h})_\Gamma \qquad \text{eq.3}$$

since $\delta^h \subset \delta$ we can also write

$$a(\vec{w}^h,\vec{u}) = (\vec{w}^h,\vec{f}) + (\vec{w}^h,\vec{h})_\Gamma \qquad \text{eq.4}$$

Subtracting eq. 4 from eq. 3 we get

$$a(\vec{w}^h,\vec{u}^h) - a(\vec{w}^h,\vec{u}) = 0$$

employing bi-linearity we have

$$a(\vec{w}^h,\vec{u}^h - \vec{u}) = a(\vec{w}^h,\vec{e}) = 0$$

and we are done with the proof of (A).

Proof of (B)
Consider

$$a(\vec{e}+\vec{w}^h,\vec{e}+\vec{w}^h) = a(\vec{e},\vec{e}) + 2a(\vec{e},\vec{w}^h) + a(\vec{w}^h,\vec{w}^h) \quad \text{eq. 5}$$

Note that by symmetry and the result to (A) we have
$a(\vec{e},\vec{w}^h) = a(\vec{w}^h,\vec{e}) = 0$.
Recalling that $a(w,w) \geq 0$ we can use eq. 5 to write

$$a(\vec{e},\vec{e}) \leq a(\vec{e}+\vec{w}^h,\vec{e}+\vec{w}^h) \quad \text{eq. 6}$$

Note that any $\vec{U}^h \in \delta^h$ can be written as $\vec{U}^h = \vec{u}^h + \vec{w}^h$
Thus we can write

$$\vec{e} + \vec{w}^h = \vec{u}^h - \vec{u} + \vec{w}^h = \vec{U}^h - \vec{u}$$

substituting this into eq. 6 we have

$$a(\vec{e},\vec{e}) \leq a(\vec{U}^h - \vec{u},\vec{U}^h - \vec{u})$$

and we are done with (B).

Proof of (C)
From (A) we have

$$a(\vec{w}^h, \vec{e}) = 0$$

Noting that $\vec{e} = \vec{u}^h - \vec{u}$ we can write $\vec{u} = \vec{u}^h - \vec{e}$ which we substitute into eq. 6 and to get

$$a(\vec{u}, \vec{u}) = a(\vec{u}^h - \vec{e}, \vec{u}^h - \vec{e}) = a(\vec{u}^h, \vec{u}^h) - 2a(\vec{u}^h, \vec{e}) + a(\vec{e}, \vec{e})$$

From (A) we have that $a(\vec{w}^h, \vec{e}) = 0$. Since $\delta^h = V^h$ can write this as $a(\vec{u}^h, \vec{e}) = 0$, thus

$$a(\vec{u}, \vec{u}) = a(\vec{u}^h, \vec{u}^h) + a(\vec{e}, \vec{e}) \quad \text{eq. 7}$$

and we are done with (C).
Since $a(\vec{e}, \vec{e}) \geq 0$ we can see from eq.7 that

$$a(\vec{u}^h, \vec{u}^h) \leq a(\vec{u}, \vec{u})$$

says the FE solution underestimates the exact energy.

We use these results in writing the finite element error. We will focus on the "Standard FE error" where we are interested in errors up to $\|\vec{e}\|_m$ (m is highest order derivative in $a(\bullet, \bullet)$) and we consider $\vec{u} \in H^r, r > m$.

Restating the convergence equation given before:
For every $\vec{u} \subset H^r$, there exists a $\vec{U}^h \in \delta^h$ and constant c, independent of $\vec{u}$ and h, such that

$$\|\vec{u} - \vec{U}^h\|_m \leq ch^\alpha \|\vec{u}\|_r \quad \text{eq.8}$$

where $\alpha = k + 1 - m$ is the convergence rate. We want to go from here to the finite element error equation for our elliptic boundary value problems which is:

$$\|\vec{e}\|_m \leq \bar{c}h^\alpha \|\vec{u}\|_r$$

Proof

$$\|\vec{e}\|_m \le \frac{1}{c_1} a(\vec{e},\vec{e})^{1/2} \quad \text{(equivalence of norms)}$$

$$\|\vec{e}\|_m \le \frac{1}{c_1} a(\vec{u}-\vec{U}^h,\vec{u}-\vec{U}^h)^{1/2} \quad \text{(best approximation theorem)}$$

$$\|\vec{e}\|_m \le \frac{c_2}{c_1}\left\|\vec{u}-\vec{U}^h\right\| \quad \text{(equivalence of norms)}$$

$$\|\vec{e}\|_m \le c\frac{c_2}{c_1} h^\alpha \|\vec{u}\|_r = \bar{c}h^\alpha \|\vec{u}\|_r \quad \text{(error equation – eq. 8)}$$

where $\alpha = k+1-m$ for $\vec{u} \in H^r, r > m$

Comment: As long as $k+1 > m$ and $r > m$ (this means that $\vec{u} \in H^{k+1}$ we have optimal convergence in the H^m norm (that is the $r-s$ term will not dictate convergence).

With some more math we can show that when $\vec{u} \in H^{k+1}$ we can write the error in lower H^s norms $0 \le s \le m$ as

$$\|\vec{e}\|_s \le ch^\beta \|\vec{u}\|_{k+1}$$

where the constant c is independent of u and h, and $\beta = \min(k+1-s, 2(k+1-m))$

Remember that we do not always have $r > m$ in which case we do not have optimal convergence since $\alpha = \min(k+1-s, r-s)$ for s^{th} norm.

Note that with this we can state the finite element error for common finite elements (Actually restate them now in terms of the finite element error).

Let δ^h be our linear ($k=1$) finite element. For $u \subset H^2$ we have

$$\|e\|_0 \le ch^2 \|u\|_2$$
$$\|e\|_1 \le ch \|u\|_2$$

Let δ^h be our quadratic ($k=2$) finite elements. For $u \subset H^3$ we have

$$\|e\|_0 \le ch^3 \|u\|_3$$
$$\|e\|_1 \le ch^2 \|u\|_3$$

Let δ^h be isoparametric elements of order k. In the case where there is no geometric approximation of the mesh to the domain ($\bar{\Omega} \equiv \bigcup_{e=1}^{n_{el}} \bar{\Omega}^e$) and only slightly curved elements. For $u \subset H^{k+1}$ we have

$$\|e\|_0 \le ch^{k+1} \|u\|_{k+1}$$
$$\|e\|_1 \le ch^k \|u\|_{k+1}$$