K is symmetric $K_{AB} = K_{BA}$

Positive Definiteness of K

Def: $[A]$, an $n \times n$ matrix is positive definite if

$\lfloor C \rfloor [A] \{C\} \geq 0 \quad \forall \{C\}$ and note- $\lfloor C \rfloor = \{C\}^T$ (a row)

$\lfloor C \rfloor [A] \{C\} = 0$ iff $\{C\}^{n\times 1} = 0$

Note: if $[A]$ is pos. def. - it has a unique inverse and its eigen values are real and positive

Proof of Pos. Def. of our $[K]$ matrices -

$$w^h = \sum_{A=1}^{n} C_A N_A = \lfloor C \rfloor \{N\} \qquad w^h \in \mathcal{V}^h$$

consider -

$$\lfloor C \rfloor [K] \{C\} = \sum_{A=1}^{N} \sum_{B=1}^{N} C_A K_{AB} C_B = \sum_{A=1}^{N} \sum_{B=1}^{N} C_A a(N_A, N_B) C_B$$

(def. of K_{AB})

$$= a\left(\sum_{A=1}^{N} C_A N_A , \sum_{B=1}^{N} C_B N_B\right) \quad \text{(bilinearity of } a(\cdot,\cdot))$$

$$= a(w^h, w^h) \quad \text{(def. of } w^h)$$

for our example problem

$$= \int_0^1 (w^h_{,x})^2 dx \geq 0 \text{ by def. of } \mathcal{V}^h$$

note for our previous form with K it is

$$\int_0^1 K (w^h_{,x})^2 dx \Leftarrow$$ we find we must say $K > 0$ to get things to work → will see again

To complete- assume

$\lfloor C \rfloor [K] \{C\} = 0$, this requires $\int_0^1 (w^h_{,x})^2 dx = 0$ on Ω

requires $w^h_{,x} = 0$ on Ω

Integrating this does allow
$w = \text{Constant}$
Thus if Constant $\neq 0$ we do not have positive definite

However, recall $w^h \in \mathcal{V}^h$

$$\mathcal{V}^h = \{ w \mid w \in H^1, \underline{\underline{w|_{\Gamma_g} = 0}} \}$$

Thus Constant $= 0$ is required -

and we are positive definite.

a

Section 1.7

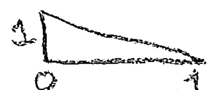
To demonstrate the process of making finite elements - lets do a simple set of example shape functions where $n=1$ and $n=2$ for our specific test problem

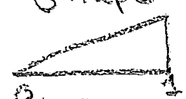
For $n=1$

$$w^h = c_1 N_1 \qquad N_1(1)=0 \text{ for } w^h \in \mathcal{V}$$

$$u^h = v^h + g^h = d_1 N_1 + g N_2 \qquad N_2(1)=1 \text{ for } u^h \in \mathcal{S}$$

Consider simple linear shape functions





$$N_1(x) = 1-x \qquad N_2(x) = x$$

$$N_{1,x} = -1 \qquad N_{2,x} = 1$$

$$\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F} \qquad \underset{\sim}{K}_{1\times1} = K_{11}, \ \underset{\sim}{d}_{1\times1} = d_1, \ \underset{\sim}{F}_{1\times1} = F$$

$$K_{AB} = a(N_A, N_B), \quad F_A = (N_A, f) + N_A(0)h - a(N_A, N_{n+1})g$$

$$K_{11} = a(N_1,N_1) = \int_0^1 N_{1,x} N_{1,x}\,dx = \int_0^1 (-1)(-1)\,dx = x\Big|_0^1 = 1$$

$$F_1 = (N_1, f) + N_1(0)h - a(N_1, N_2)g$$
$$= \int_0^1 N_1 f(x)\,dx + N_1(0)h - \int_0^1 N_{1,x} N_{2,x}\,dx \; g$$
$$= \int_0^1 (1-x) f(x)\,dx + h - \int_0^1 (-1)(1)\,dx \; g = \int_0^1 (1-x)f(x)\,dx + h + g$$

$$d_1 = K_{11}^{-1} F_1 = \frac{1}{1}\left(\int_0^1 (1-x) f(x)\,dx + h + g \right) = F_1$$

$$u^h = v^h + g^h = d_1 N_1 + g N_2 = \overbrace{(1-x)}^{N_1}\left[\int_0^1 (1-y) f(y)\,dy + h + g\right] + g\,x$$

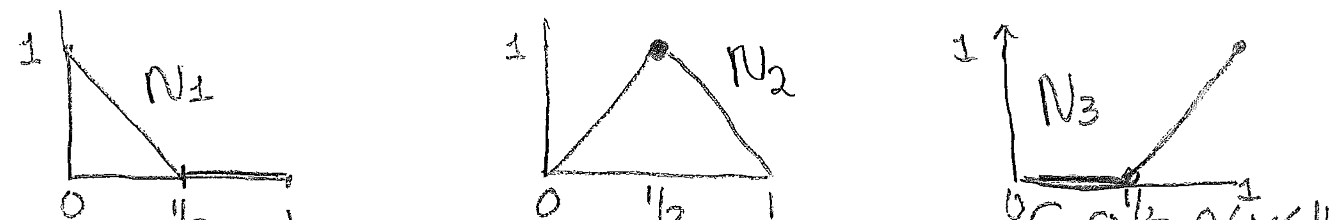
Lets Consider $n=2$ (2 dof) case using two sets of shape functions

$$[K]=\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix},\quad \{d\}=\begin{Bmatrix}d_1\\ d_2\end{Bmatrix},\quad \{F\}=\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$K_{AB}=a(N_A,N_B)=\int_0^1 N_{A,x}\,N_{B,x}\,dx$$

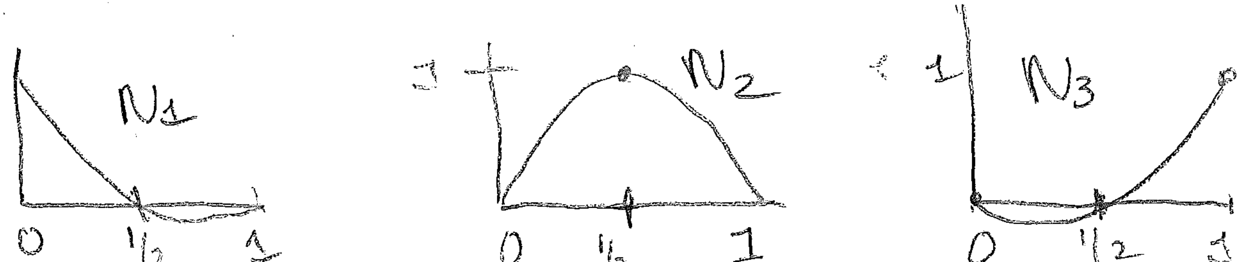
$$F_A=(N_A,f)+N_A(0)\,h-a(N_A,N_3)\,g=\int_0^1 N_A f\,dx+N_A(0)\,h-\int_0^1 N_{A,x}\,N_{3,x}\,dx\,g$$

Shape function Set 1 - Piecewise Linear



$$N_1=\begin{cases}1-2x & 0\le x\le 1/2\\ 0 & 1/2\le x\le 1\end{cases}\qquad N_2=\begin{cases}2x & 0\le x\le 1/2\\ 2(1-x) & 1/2\le x\le 1\end{cases}\qquad N_3=\begin{cases}0 & 0\le x\le 1/2\\ 2x-1 & \frac{1}{2}\le x\le 1\end{cases}$$

Shape function Set 2 - Quadratic



$N_1=1-3x+2x^2$ $\quad N_2=4x(1-x)$ $\quad N_3=x(2x-1)$

C

Some Terms for Piecewise linear shape functions

$$K_{12} = \int_0^1 N_{1,x} N_{2,x} dx = \int_0^{1/2} (-2)\, 2\, dx + \int_{1/2}^1 (0) -2\, dx = -4x\Big|_0^{1/2} = -2$$

$$K = 2\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$F_1 = \int_0^1 N_1 f dx + N_1(0) h - \int_0^1 N_{1,x} N_{3,x}\, dx\, g$$

$$= \int_0^{1/2} (1-2x) f dx + \int_{1/2}^1 (0) f dx + h - \int_0^{1/2} -2(0)\, dx\, g - \int_{1/2}^1 0\; 2\, dx\, g$$

$$F_1 = \int_0^{1/2} (1-2x) f\, dx + h$$

$$F_2 = 2\int_0^{1/2} x f dx + 2\int_{1/2}^1 (1-x) f dx + 2g$$

Some terms for the Quadratic shape function

$N_{1,x} = 4x-3$, $N_{2,x} = 4-8x$, $N_{3,x} = 4x-1$

$$K_{11} = \int_0^1 N_{1,x} N_{1,x} dx = \int_0^1 (4x-3)(4x-3) dx = \left[\tfrac{16}{3}x^3 - 12x^2 + 9x\right]_0^1 = 7/3$$

$$K_{12} = \int_0^1 N_{1,x} N_{2,x}\, dx = \int_0^1 (4x-3)(4-8x) dx = -8/3$$

$$K_{22} = \int_0^1 N_{2,x} N_{2,x}\, dx = \int_0^1 (4-8x)(4-8x) dx = 16/3$$

$$[K] = \frac{1}{3}\begin{bmatrix} 7 & -8 \\ -8 & 16 \end{bmatrix}$$

$$F_1 = \int_0^1 N_1 f dx + N_1(0) h - \int_0^1 N_{1,x} N_{3,x}\, dx\, g$$

$$F_1 = \int_0^1 (1-3x+2x^2) f dx + h - \int_0^1 (4x-3)(4x-1)\, dx\, g$$

$$F_1 = \int_0^1 (1-3x+2x^2) f dx + h - g/3$$

$$F_2 = \int_0^1 (4x-4x^2) f dx + 8/3\, g$$

Recall the exact solution

$$u = g + (1-x)h + \int_x^1 \left\{ \int_0^y f(z)\,dz \right\} dy_1$$

Consider cases

$$\begin{aligned} f&=0 & u &= g + (1-x)h \\ f&=p & u &= g + (1-x)h + \tfrac{p}{2}(1-x^2) \\ f&=qx & u &= g + (1-x)h + \tfrac{q}{6}(1-x^3) \end{aligned}$$

How do our F.E. Solutions compare?

1 DOF

$$u^h = v^h + g^h = N_1 d_1 + N_2 d_2$$

From before

$$u^h = (1-x)\left(\int_0^1 (1-y) f\,dy + h + g\right) + xg$$

for $f=0$ $\quad u^h = (1-x)h + g \quad$ (exact)

for $f=p$

$$u^h = (1-x)\left(\int_0^1 (1-y)p\,dy + h + g\right) + gx$$

$$u^h = (1-x)h + g + \frac{p}{2}(1-x)$$

approximate — exact is $\frac{p}{2}(1-x^2)$

for $f = qx$

$$u^h = (1-x)h + g + \frac{q}{6}(1-x)$$

exact is $(1-x^3)$

2DOF cases

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{bmatrix} K \end{bmatrix}^{-1} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$u^h = d_1 N_1 + d_2 N_2 + g N_3$$

2DOF Linear

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \int_0^{1/2} (1-2x) f \, dx + h \\ 2\int_0^{1/2} x f dx + 2\int_{1/2}^{1} (1-x) f dx + 2g \end{Bmatrix}$$

$$d_1 = \int_0^{1/2} (1-2x) f \, dx + h + \int_0^{1/2} x f dx + \int_{1/2}^{1} (1-x) f dx + g$$

$$d_2 = \frac{1}{2}\int_0^{1/2} (1-2x) f dx + \frac{h}{2} + \int_0^{1/2} x f dx + \int_{1/2}^{1} (1-x) f dx + g$$

for $f=0$

$u^h = g + h(1-x)$ exact again

for $f=p$

$$u^h = g + h(1-x) + \frac{p}{2} N_1 + \frac{3p}{8} N_2$$

$$u^h = g + h(1-x) + \begin{cases} \frac{p}{2}(1-2x) + \frac{3p}{4} x & 0 \le x \le 1/2 \\ \frac{3p}{4}(1-x) & 1/2 \le x \le 1 \end{cases}$$

for $f = qx$

$$u = g + h(1-x) + q/6 \, N_1 + \frac{7q}{48} N_2$$

$$u^h = g + h(1-x) + \begin{cases} q/6 (1-2x) + \frac{7q}{24} x & 0 \le x \le 1/2 \\ 7q/48 (1-x) & 1/2 \le x \le 1 \end{cases}$$

for $f=p$ and $f=qx$ solution is two linear segments that are only approximate solutions

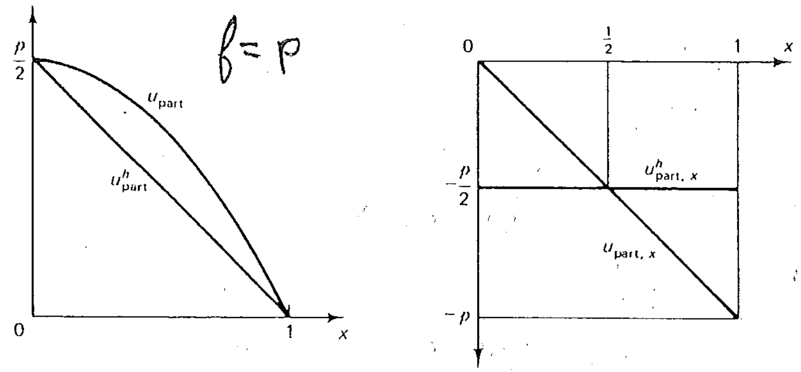


Figure 1.7.3 Comparison of exact and Galerkin particular solutions, Example 1, case (ii).

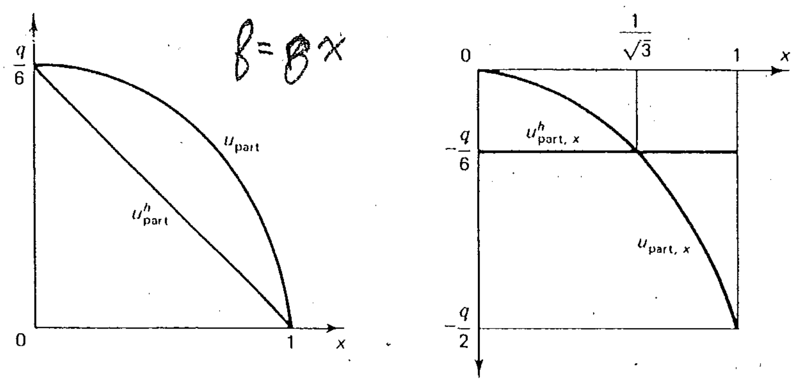


Figure 1.7.4 Comparison of exact and Galerkin particular solutions, Example 1, case (iii).

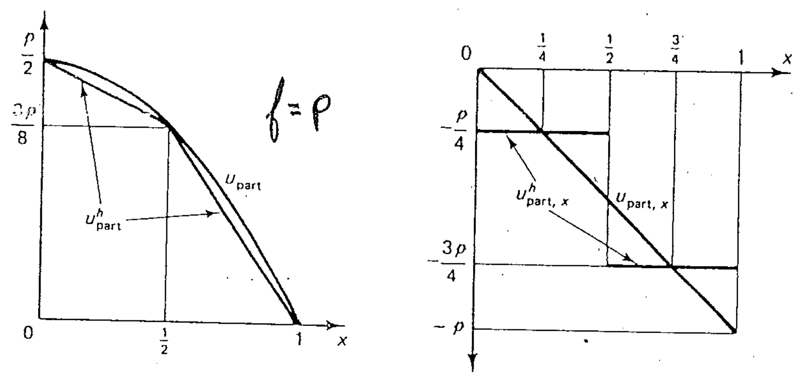


Figure 1.7.7 Comparison of exact and Galerkin particular solutions, Example 2, case (ii).

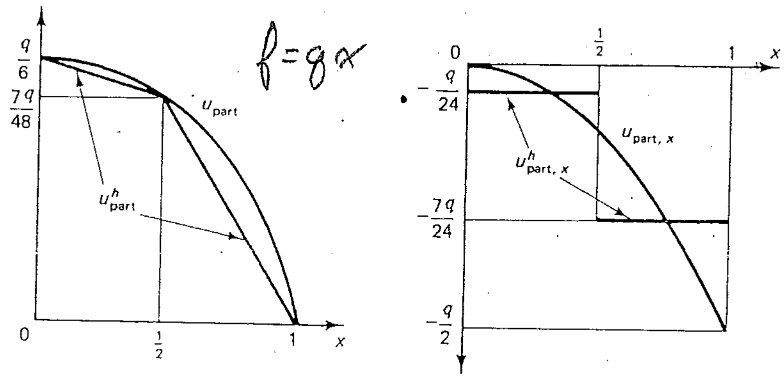


Figure 1.7.8 Comparison of exact and Galerkin particular solutions, Example 2, case (iii).

2 DOF Quadratic

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \frac{1}{16}\begin{bmatrix} 16 & 8 \\ 8 & 7 \end{bmatrix} \begin{Bmatrix} \int_0^1 (1-3x+2x^2)\, f\, dx + h + g/3 \\ \int_0^1 (4x-4x^2)\, f\, dx + 8/3\, g \end{Bmatrix}$$

for $f=0$

$d_1 = h+g$, $d_2 = \frac{h}{2}+g$

$$u^h = \underbrace{(h+g)}_{d_1}\underbrace{(1-3x+2x^2)}_{N_1} + \underbrace{\left(\frac{h}{2}+g\right)}_{d_2}\underbrace{4x(1-x)}_{N_2} + \underbrace{g}_{g}\underbrace{(x(2x-1))}_{N_3}$$

$u^h = h(1-x)+g$ ⇐ the correct linear

for $f=p$

$d_1 = h+g+p/2$, $d_2 = \frac{h}{2}+g+3p/8$

After doing $u^h = d_1N_1 + d_2N_2 + gN_3$ you get

$u^h = h(1-x)+g+p/2\,(1-x^2)$ ⇐ The exact solution

for $f = qx$

You will get $d_1 = h+g+q/6$, $d_2 = \frac{h}{2}+g+7q/48$

$u^h = h(1-x)+g+\frac{q}{12}(2+x-3x^2)$ approximate

⇒ If F.E. Space is rich enough to give the exact solution – it <u>will</u>

⇒ In 1D nodal values are exact – that only happens in 1D

⇒ In 1D case will get derivative correct somewhere in each element – <u>only 1D</u>