Appendix 4.I.1

Linear space – a collection of objects that satisfy the following: If u and v are members of a linear space and α and β are scalars, then $\alpha u + \beta v$ is also a member of that linear space.

This applies to vector objects $\vec{u}$ and $\vec{v}$ where now addition is by component

$$\vec{u} + \vec{v} = (u_1, u_2, u_3, \ldots u_n) + (v_1, v_2, v_3, \ldots v_n) =$$

$$(u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots u_n + v_n)$$

and scalar multiplication means

$$\alpha\vec{u} = (\alpha u_1, \alpha u_2, \alpha u_3, \ldots \alpha u_n)$$

Thus if $\vec{u}$ and $\vec{v}$ are members of a linear space and α and β are scalars, then $\alpha\vec{u} + \beta\vec{v}$ is also a member of that linear space where

$$\alpha\vec{u} + \beta\vec{v} = (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \alpha u_3 + \beta v_3, \ldots \alpha u_n + \beta v_n)$$

Linear spaces have very nice properties that make it easy for us to “prove” things will behave the way we would like. Thus we want to be sure to know when the contributions to our FE weak forms are members of linear spaces. (For this class they will be, as you get to more complex problems they may not be, then things you have to figure out what you can use.

Key linear space properties we like to employ are inner products (like our integrals to be inner products) and norms (which will represent a measure of size).

Definition: An inner product $\langle \bullet,\bullet \rangle$ on a real linear space A is a map $\langle \bullet,\bullet \rangle : A \times A \rightarrow \Re$ with the following properties:

i) $\langle u,v \rangle = \langle v,u \rangle$ (symmetry)

ii) $\langle \alpha u,v \rangle = \alpha \langle u,v \rangle$

iii) $\langle u+v,w \rangle = \langle u,w \rangle + \langle v,w \rangle$ (ii) and iii) are linearity)

iv) $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0$ if and only if $u = 0$

(positive definiteness)

Definition: Let $\{A, \langle \bullet,\bullet \rangle\}$ be an inner produce space (i.e., a linear space A with and inner product $\langle \bullet, \ \bullet \rangle$ defined on A. Then $u,v \in A$ are said to be orthogonal (with respect to $\langle \bullet,\bullet \rangle$) if $\langle u,v \rangle = 0$

Schwartz inequality $\langle u,v \rangle^2 \leq \langle u,u \rangle \langle v,v \rangle$

Definition: A norm $\|\bullet\|$ on a linear space A is a map $\|\bullet\| : A \rightarrow \Re$, with the following properties:

Let $u,v \in A$ and $\alpha \in \Re$, then

i) $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$ (pos. def.)

ii) $\|\alpha u\| = |\alpha| \|u\|$

iii) $\|u+v\| \leq \|u\| + \|v\|$ triangle inequality

$\{A, \langle \bullet,\bullet \rangle\}$ possesses a natural norm - $\|u\| = \langle u,u \rangle^{1/2}$

Some time we do not have a norm, but do have a semi-norm.

Definition: A semi-norm $|\bullet|$ on a linear space A is a map $|\bullet|: A \rightarrow \Re$, with the following properties:

Let $u, v \in A$ and $\alpha \in \Re$, then

i) $|u| \geq 0$ (positive semidefinite)

ii) $|\alpha u| = |\alpha||u|$

iii) $|u + v| \leq |u| + |v|$ (triangle inequality)

A start at defining Sobolev inner products and norms:

Consider a domain $\Omega \subset \Re^{n_{sd}}$, $n_{sd} \geq 1$ (will be the spatial dimension – 1D, 2D, 3D), and let $u, v: \Omega \rightarrow \Re$

The $L_2(\Omega)$ (or equivalently $H^o(\Omega)$) inner product and norm are defined by

$$(u, v) = (u, v)_0 = \int_\Omega uv \, d\Omega$$

$$\|u\| = (u, u)^{1/2}$$

The $H^1(\Omega)$ inner product and norm are defined by

$(u, v)_1 = \int_\Omega (uv + u_{,i} v_{,i}) \, d\Omega$ (sum $1 \leq i \leq n_{sd}$)

$$\|u\| = (u, u)_1^{1/2}$$

Note on Notation – as we go forward the summation on repeated subscripts from 1 to whatever will always be assumed (unless specifically stated otherwise)

Thus $u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 + \ldots u_n v_n$. Most of the time $n = n_{sd}$. Some time it will be something like the number of elements (n_{el}) or something that is a function of the mesh size.

Note – See 4.I.2 to see the further definition of the Sobolev spaces.

A key thing we need to do is to understand what can be used for the weighting functions, w, and trial functions, u. Sobolev spaces and their associate properties are very useful here.

$$u \in H^0 \text{ if } \int_\Omega u^2 d\Omega < \infty$$

$$u \in H^1 \text{ if } \int_\Omega (uu + u_{,i} u_{,i}) d\Omega < \infty$$

Recall for our problem we had $f : \overline{\Omega} \rightarrow \Re$, $\kappa \in \Re$, $\kappa > 0$ (and finite) and had an integral that looked like $\int_0^1 w_{,x} \kappa u_{,x} dx$. It is clear that this integral will be well behaved if $u \in H^1$ and $w \in H^1$.

Thus we will want to use this when we qualify the spaces (the possible places) we select w and u from.

We will want our weighting function space to be:

$$V = \{w \mid w \in H^1, w|_{\Gamma_g} = 0\}$$

In words: V is the set of functions w that satisfy the conditions to the right of the vertical bar that are $w \in H^1$ and w on the portion of the boundary we have essential boundary conditions (BC) is zero.

We will want our trial function space to be:

$$\delta = \{u \mid u \in H^1, u|_{\Gamma_g} = g\}$$

Questions: Is V a linear space? Is δ a linear space?

We will find it convenient to write our weak form using a bit of abstraction in the equations.

A restatement of the weak form is:

Given had $f : \overline{\Omega} \rightarrow \Re$, $\kappa \in \Re$, $\kappa > 0$, and constants g and h, find $u \in \delta$ such that for all $w \in V$

$$a\left(w,u\right) = \left(w,f\right) + \left(w,h\right)_\Gamma$$

for the problem we have thus far we have:

$$a\left(w,u\right) = \int_0^1 w_{,x} \kappa u_{,x}\, dx$$

$$\left(w,f\right) = \int_0^1 wf\, dx$$

$$\left(w,h\right)_\Gamma = w(0)h$$

We can check the symmetry and bilinearity of the $a\left(w,u\right)$ and $\left(w,f\right)$