Equivalence of strong (S) and weak (W) forms

We want to demonstrate that:

- If u is a solution to (S) it is a solution to (W)
- If u is a solution to (W) it is a solution to (S)
- The solution u is unique

For simplicity we will use $\kappa = 1$ and the specific BCs $u(1) = g$ and $-u_x(0) = h$

Given u is a solution to (S) we have

(S) $u_{,xx} + f = 0$ in Ω, $u(1) = g$ and $-u_x(0) = h$

Therefore we can write

$$-\int_0^1 w\left(u_{,xx} + f\right)dx = 0 \;\; \forall w \in V \text{ (}\forall\text{ means “for all”)}$$

Integrate the first term by parts to get

$$\int_0^1 w_{,x} u_{,x} dx - \int_0^1 wf\,dx - wu_{,x}\,|_0^1 = 0 \;\; \forall w \in V$$

note: $w(1) = 0$ ($w \in V$), and $-u_{,x}(0) = h$

$$\int_0^1 w_{,x} u_{,x} dx = \int_0^1 wf\,dx + w(0)h \;\; \forall w \in V$$

recalling that $u \in \delta$ (which requires $u(1) = g$) we are set (as expected since it what we did to begin with)

Now the next one of given u is a solution to (W) it is a solution to (S) actually has a bit more to it.

Given u is a solution to (W) we have $u \in \delta$ (thus we have already taken care of the essential BC $u(1) = g$)

(W) $\int_0^1 w_{,x} u_{,x} dx = \int_0^1 wf\, dx + w(0)h \ \ \forall w \in V$

Integrate the first by parts (go the other way) we get

$$\int_0^1 w(u_{,xx} + f)dx - wu_{,x} |_0^1 + w(0)h = 0 \ \ \forall w \in V$$

note: $w(1) = 0$ ($w \in V$) so this reduces to

$$\int_0^1 w(u_{,xx} + f)dx + w(0)(u_{,x}(0) + h) = 0 \ \ \forall w \in V \quad \text{eq. A}$$

note that this says the sum of two things has to be zero but we need to still show that the two pieces of $u_{,xx} + f = 0$ and $-u_x(0) = h$ are set. Since we need to meet eq. A $\forall w \in V$ we can select specific ones that let us see one piece of the equation at a time.

Select $w = \phi(u_{,xx} + f)$ where ϕ is a smooth positive function that is 0 at both ends ($\phi(x) > 0, 0 < x < 1$ and $\phi(0) = \phi(1) = 0$). With that eq. A becomes

$$\int_0^1 \phi(u_{,xx} + f)^2 dx + 0(u(0) + h) = 0$$

With $\phi(x) > 0$ in the domain with $(u_{,xx} + f)^2 \geq 0$ for $0 < x < 1$ we have no option but to have

$(u_{,xx} + f)^2 = 0$ for $0 < x < 1$ which is the domain Ω

With this eq. A is

$$0 + w(0)(u_{,x}(0) + h) = 0 \ \ \forall w \in V$$

Since $\forall w \in V$ puts no restriction on $w(0)$ then

$$-u_x(0) = h$$

And we are done.

To show the solution is unique go from weak form
Given …. , find $u \in \delta$ such that $\forall w \in V$

$$a\left(w,u\right) = \left(w,f\right) + \left(w,h\right)_{\Gamma}$$

Lets assume two solutions, $u1$ and $u2$ (will show it is not possible). Then

$$a\left(w,u1\right) = \left(w,f\right) + \left(w,h\right)_{\Gamma} \quad \text{eq. B}$$

$$a\left(w,u2\right) = \left(w,f\right) + \left(w,h\right)_{\Gamma} \quad \text{eq. C}$$

(eq. B) – (eq. C) yields

$$a\left(w,u1\right) - a\left(w,u2\right) = 0$$

which by bi-linearity is

$$a\left(w,(u1-u2)\right) = 0$$

select $w = u1 - u2$ note $w \in V$ giving us

$$a\left((u1-u2),(u1-u2)\right) = 0$$

By the requirement of positive definiteness of $a(\bullet,\bullet)$ the only way we can have meet this is for

$$u1 - u2 = 0 \text{ or } u1 = u2$$

thus the solution is unique.

We are now ready to go the next step – going from the infinite dimensional case of the exact solution to what will ultimately lead to the desired finite element form.

For our finite dimensional finite element form we typically use a superscript h

For the exact solution we have infinite dimensional spaces V and δ. We define finite dimensional spaces

$$V^h \subset V \text{ and } \delta^h \subset \delta$$

from which take finite dimensional w^h and u^h

$$w^h \in V^h \text{ and } u^h \in \delta^h$$

recalling that spaces V and δ (and thus V^h and δ^h) differ by only by how they deal with the essential boundary conditions, and the fact that linear spaces have properties we like, we will decompose u^h into

$$u^h = v^h + g^h \text{ where } v^h \in V^h \text{ and } g^h \in \delta^h$$

that is g^h will satisfy the essential BC and v^h will satisfy the homogeneous version of the BC (ie. are 0).

We will select to use the same form for w^h and v^h which will be to define V^h as a linear combination of n given functions $N_A : \Omega \to \Re \;\; A = 1(1)n$ (A starts at 1 and goes to n by increments of 1). With this

$$w^h = C_A N_A = \sum_{A=1}^{n} C_A N_A = C_1 N_1 + C_2 N_2 + C_3 N_3 + \ldots + C_n N_n$$

$$v^h = d_A N_A$$

u^h needs to add in terms for essential BC

$$u^h = v^h + g^h = \sum_{A=1}^{n} d_A N_A + \sum_{B=n+1}^{n+m} g_B N_B$$

where m is the number of shape functions need to cover the essential BC. For the text example $m = 1$

$$u^h = v^h + g^h = \sum_{1}^{n} d_A N_A + g N_{n+1}$$

Substituting into abstract form and using bi-linearity, which says $a\big(w,(v+g)\big)=a\big(w,v\big)+a\big(w,g\big)$, we have

$$a\big(w,v\big)=\big(w,f\big)+\big(w,h\big)_\Gamma-a\big(w,g\big)$$

Putting in the summations we have

$$a\left(\sum_{A=1}^{n}C_AN_A,\sum_{B=1}^{n}d_BN_B\right)=\left(\sum_{1}^{n}C_AN_A,f\right)+\left(\sum_{1}^{n}C_AN_A,h\right)_\Gamma$$

$$-a\left(\sum_{A=1}^{n}C_AN_A,\sum_{B=n+1}^{n+m}g_BN_B\right) \text{ eq. D}$$

which for the specific textbook case is

$$a\left(\sum_{A=1}^{n}C_AN_A,\sum_{B=1}^{n}d_BN_B\right)=\left(\sum_{1}^{n}C_AN_A,f\right)+\sum_{1}^{n}C_AN_A(0)h$$

$$-a\left(\sum_{A=1}^{n}C_AN_A,gN_{n+1}\right)$$

note $a\left(\sum_{A=1}^{n}w_A,v\right)=\sum_{A=1}^{n}a\big(w_A,v\big)$,

$a\left(\sum_{A=1}^{n}w_A,\sum_{B=1}^{m}v_B\right)=\sum_{A=1}^{n}\sum_{B=1}^{m}a\big(w_A,v_B\big)$ thus eq. D is

$$\sum_{A=1}^{n}\sum_{B=1}^{n}a\big(C_AN_A,d_BN_B\big)=\sum_{A=1}^{n}\big(C_AN_A,f\big)+\sum_{A=1}^{n}\big(C_AN_A,h\big)_\Gamma$$

$$-\sum_{A=1}^{n}\sum_{B=n+1}^{n+m}a\big(C_AN_A,g_BN_B\big)$$

Note that all terms have $\sum_{A=1}^{n}$ which can pulled out

$$\sum_{A=1}^{n} C_A \left[\sum_{B=1}^{n} a\left(N_A, N_B\right) d_B - \left(N_A, f\right) - \left(N_A, h\right)_\Gamma + \sum_{B=n+1}^{n+m} a\left(N_A, N_B\right) g_B \right] = 0$$

Lets define everything in the [ ] as G_A then we have

$$\sum_{A=1}^{n} C_A G_A = 0$$

The only way to ensure this is that $G_A = 0,\ A = 1(1)n$.
This gives us our finite element system, which is:

$$\sum_{B=1}^{n} a\left(N_A, N_B\right) d_B = \left(N_A, f\right) + \left(N_A, h\right)_\Gamma - \sum_{B=n+1}^{n+m} a\left(N_A, N_B\right) g_B \quad \text{eq. E}$$

$$A = 1(1)n$$

for the text book case this is:

$$\sum_{B=1}^{n} a\left(N_A, N_B\right) d_B = \left(N_A, f\right) + N_A(0), h - a\left(N_A, N_{n+1}\right) g$$

$$A = 1(1)n$$

Define $K_{AB} = a\left(N_A, N_B\right)$,

$$F_A = \left(N_A, f\right) + \left(N_A, h\right)_\Gamma - \sum_{B=n+1}^{n+m} a(N_A, N_B) g_B$$

Using these in eq. E we have

$$\sum_{B=1}^{n} K_{AB} d_B = F_A,\ A = 1(1)n$$

Which is n equations in n unknowns or if you will is

$$\left[K\right]_{nxn} \{d\}_{nx1} = \{F\}_{nx1}$$

and we have our FE all the way to matrix algebra.
In summary we have done (S)$\Leftrightarrow$(W)$\approx$(G) $\Rightarrow$(M)