Will cover text sections 1.2 to 1.10 and 1.12 to 1.16 will also cover Appendix 4.I.1

1D ODE

$\left(\kappa u_{,x}\right)_{,x} + f = 0$ in Ω

u - the dependent variable
f - forcing function
κ - material parameter
Ω - domain

$_{,x} = \dfrac{d}{dx} \quad \left(\kappa u_{,x}\right)_{,x} = \dfrac{d}{dx}\left(\kappa \dfrac{du}{dx}\right)$ if κ constant $\left(\kappa u_{,x}\right)_{,x} = \kappa u_{,xx} = \kappa \dfrac{d^2 u}{dx^2}$

In text $\kappa = 1$

In a 1D domain from *a* to *b*
$\Omega =]a,b[$, $a < x < b$ does not include the boundary

The closure includes the boundary
$\overline{\Omega} = [a,b]$, $a \le x \le b$

The boundary in 1D is the ends $x = a,b$

$f \in \Re$, $\kappa \in \Re$, $\kappa > 0$

To properly specific the problem we also need boundary conditions (BCs) at a and b which can be either essential (Dirichlet) or natural (Neumann). The must be minimum number of essential BC. In the 1D case need an essential BC at a or b. Can have essential at both. Can not have natural at both. (Each end has only one BC for the second order ODE.)

Essential BC

$u = g$ on Γ_g for example $u(1) = g$

Natural BC

$-\kappa u_{,x} = h$ on Γ_h for example $-\kappa(0)u_{,x}(0) = h$

We can now state the strong form of the problem

Given $f : \overline{\Omega} \to \Re$, $\kappa \in \Re$, $\kappa > 0$ and constants g and h, find $u : \overline{\Omega} \to \Re$ such that

$$\left(\kappa u_{,x}\right)_{,x} + f = 0 \quad \text{in } \Omega$$

$$u = g \text{ on } \Gamma_g \text{ for example } u(1) = g$$

$$-\kappa u_{,x} = h \text{ on } \Gamma_h \text{ for example } -\kappa(0)u_{,x}(0) = h$$

For the case where $\kappa = 1$, the exact solution is

$$u(x) = g + (1 - x)h + \int_x^1 \left[\int_0^y f(z)\,dz \right] dy$$

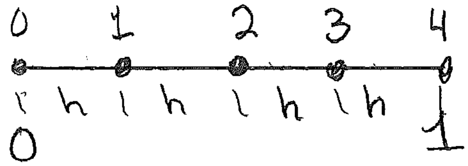
Finite Difference Example

$$\frac{d^2u}{dx^2} + f = 0 \qquad 0 < x < 1 \qquad \text{— (a)}$$

$$u(0) = g, \quad u(1) = \hat{g}$$

Central Difference Approximation

$$\frac{d^2u(x)}{dx^2} \approx \frac{d^2 u_j}{dx^2} = \frac{1}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right)$$



5 points $\quad h = \frac{1}{4} = 0.25$

Apply to eq. (a) and we get 3 equations in 3 unknowns

$$\frac{1}{h^2}\left(u_{j-1} - 2u_j + u_{j+1}\right) = f(x_j) \quad j = 1, 2, 3$$

(recall we know $u(0) = g$ and $u(1) = \hat{g}$

$$\frac{1}{h^2}(u_0 - 2u_1 + u_2) = -f(x_1) = \frac{1}{h^2}(g - 2u_1 + u_2)$$

$$\frac{1}{h^2}(u_1 - 2u_2 + u_3) = -f(x_2) = \frac{1}{h^2}(u_1 - 2u_2 + u_3)$$

$$\frac{1}{h^2}(u_2 - 2u_3 + u_4) = -f(x_3) = \frac{1}{h^2}(u_2 - 2u_3 + \hat{g})$$

Final algebraic system (multiplied all equations by -1)

$$[A]\{U\} = \{F\} \quad [A] = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \{F\} = \begin{Bmatrix} F(x_1) + g/h^2 \\ F(x_2) \\ F(x_3) + \hat{g}/h^2 \end{Bmatrix}$$

For $f(x) = 0$, $g = g$, $\hat{g} = 0$

get correct exact solution

$$u_1 = \tfrac{3}{4}g, \quad u_2 = \tfrac{1}{2}g, \quad u_3 = \tfrac{1}{4}g$$

For the problems we are interested in we will typically not be able to find an exact solution and thus need a method to get an approximate solution through some form of discretization.

The process of discretization means we will go from looking for a solution in an infinite dimensional space to looking for one in a finite dimensional space.

For finite difference we directly discretize the differential operators in the strong form. For finite elements we will discretize an alternative, but equivalent form, referred to as the weak form. One way to construct a weak form is the method of weighted residuals (more on this later).

For the current problem on a domain that goes from 0 to 1 (a =0, b =1) we will do this as follows:

$$\int_0^1 w\left[\left(\kappa u_{,x}\right)_{,x} + f\right]dx = 0$$

We could proceed with this weak form directly so long as all to functions for u we consider satisfy both BC a-priori. We can also do integration by parts ($\int_a^b uv_{,x}dx = uv\Big|_a^b - \int_a^b u_{,x}vdx$) to get

$$w\kappa u_{,x}\Big|_0^1 - \int_0^1 w_{,x}\kappa u_{,x}dx + \int_0^1 wf\,dx = 0$$

This does two things – we no longer have to take a second derivative and, as we will see, we have a way to account for the natural boundary conditions. We will still be required to satisfy the essential BC a priori. Note that since we will be required to take care of the essential BC a priori we, will be allowed to set $w = 0$ at those locations.

Integration by parts-

Basics

$$(uv)_{,x} = u_{,x} v + u v_{,x}$$

Thus -

$$\int_a^b (uv)_{,x}\, dx = \int_a^b (u_{,x} v + u v_{,x})\, dx = \int_a^b (u_{,x} v)\, dx + \int_a^b (u v_{,x})\, dx \quad \text{—} (1)$$

$$\int_a^b (uv)_{,x}\, dx = (uv)\Big|_a - (uv)\Big|_b \quad \text{—} (2)$$

Sub (2) into (1)

$$(uv)\Big|_a - (uv)\Big|_b = \int_a^b (u_{,x} v)\, dx + \int_a^b (u v_{,x})\, dx$$

solving for $\int (u v_{,x})\, dx$ we have

$$\int_a^b (u v_{,x})\, dx = \left[(uv)\Big|_a - (uv)\Big|_b \right] - \int_a^b u_{,x} v\, dx$$

$$(uv)\Big|_a^b$$

To be more specific for our current problem this will allow us to deal with the $w\kappa u_{,x}\Big|_0^1$ term as follows: Set $w(1)=0$ and note that $-\kappa(0)u_{,x}(0)=h$, thus

$$w\kappa u_{,x}\Big|_0^1 = -\left(w(1)\kappa(1)u(1)_{,x} - w(0)\kappa(0)u(0)_{,x}\right) = w(0)h$$

We can now state the weak form of the problem

Weak form:

Given $f:\overline{\Omega}\rightarrow\Re$, $\kappa\in\Re$, $\kappa>0$ and constants g and h, and $u(1)=g$, find $u:\overline{\Omega}\rightarrow\Re$ such that

$$\int_0^1 w_{,x}\kappa u_{,x}dx = \int_0^1 wf\,dx + w(0)h\Big|$$

for all “smooth” w. We will be more specific on what we mean by smooth later.

$$\int_0^1 w\left((K u_{,x})_{,x} + f\right) dx = 0$$

$$\int_0^1 w (K u_{,x})_{,x}\, dx + \int_0^1 w f\, dx = 0$$

$$w K u_{,x} \Big|_0^1 - \int_0^1 w_{,x} K u_{,x}\, dx + \int_0^1 w f\, dx = 0$$

$$w(1)\, K(1) u_{,x}(1) - w(0)\, K(0) u_{,x}(0) - \int_0^1 w_{,x} K u_{,x}\, dx + \int_0^1 w f\, dx = 0$$

→ Note: $w(1) = 0 \leftarrow \underline{u(1) = g}$

$-K(0) u_{,x}(0) = h$

$$+ w(0)\, \overset{h}{\cancel{K(0) u_{,x}(0)}} - \int_0^1 w_{,x} K u_{,x}\, dx + \int_0^1 w f\, dx = 0$$

$$\int_0^1 w_{,x} K u_{,x}\, dx = \int_0^1 w f\, dx + w(0)\, K(0) u_{,x}(0)$$

Table 2.4.1 Some examples of engineering problems in which the second-order equation (2.4.1a) and its boundary conditions (2.4.1b) arise.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)=f \quad \text{for } 0<x<L; \quad u(0)=u_0; \quad \left(a\frac{du}{dx}\right)_{x=L}=Q_L$$

Field	Primary variable u	Coefficient* a	Source term f	Secondary variable Q_0
1. Cables	Transverse deflection	T	Distributed vertical force	Axial force
2. Bars	Longitudinal displacement	EA	Distributed axial force	Axial load
3. Heat transfer	Temperature	k	Internal heat generation	Heat flux
4. Pipe flow	Hydrostatic pressure	$\frac{\pi D^4}{128\mu}$	Flow source	Flow rate
5. Viscous flows	Velocity	μ	Pressure gradient	Stress
6. Seepage	Fluid head	ε	Fluid flux	Flow
7. Electrostatics	Electrical potential	ϵ	Charge density	Electric flux

*E = Young's modulus; A = area of cross section; D = diameter of the pipe; k = thermal conductivity; μ = viscosity; T = tension; ε = permeability; and ϵ = dielectric constant.

Table 8.1.1 Some examples of the Poisson equation

$$-\nabla \cdot (k\nabla u) = f \text{ in } \Omega$$

Natural boundary condition: $k\dfrac{\partial u}{\partial n} + \beta(u - u_\infty) = q$ on Γ_q

Essential boundary condition: $u = \hat{u}$ on Γ_u

Field of application	Primary variable u	Material constant k	Source variable f	Secondary variables q, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$
Heat transfer	Temperature T	Conductivity k	Heat source g	Heat flow due to conduction $k\frac{\partial T}{\partial n}$ convection $h(T - T_\infty)$
Irrotational flow of an ideal fluid	Stream function ψ	Density ρ	Mass production σ	Velocities $\frac{\partial \psi}{\partial x} = -v$ $\frac{\partial \psi}{\partial y} = u$
	Velocity potential ϕ	Density ρ	Mass production σ	$\frac{\partial \phi}{\partial x} = u$ $\frac{\partial \phi}{\partial y} = v$
Groundwater flow	Piezometric head ϕ	Permeability K	Recharge f (pumping, $-f$)	Seepage $q = k\frac{\partial \phi}{\partial n}$ Velocities $u = -k\frac{\partial \phi}{\partial x}$ $v = -k\frac{\partial \phi}{\partial y}$
Torsion of cylindrical members	Stress function Ψ	$k = 1$ $G =$ shear modulus	$f = 2$ $\theta =$ angle of twist per unit length	$G\theta\frac{\partial \Psi}{\partial x} = -\sigma_{yz}$ $G\theta\frac{\partial \Psi}{\partial y} = \sigma_{xz}$
Electrostatics	Scalar potential ϕ	Dielectric constant ϵ	Charge density ρ	Displacement flux density D_n
Magnetostatics	Magnetic potential ϕ	Permeability μ	Charge density ρ	Magnetic flux density B_n
Membranes	Transverse deflection u	Tension in membrane T	Transversely distributed load	Normal force q