MWR Example

(S)

$$(\alpha u_{,x})_{,x} - \beta u_{,x} + f = 0 \quad \text{in } \Omega$$

$$\Omega: 0 \le x \le 1$$

$$u(0) = g$$

$$-\alpha u_{,x}(1) = h$$

Construct a weak form

$$\int_\Omega w\left[(\alpha u_{,x})_{,x} - \beta u_{,x} + f\right]dx = 0 \quad \forall w \in \mathcal{V}$$

Look at three terms

$$\int_\Omega w\,(\alpha u_{,x})_{,x}\,dx - \int_\Omega w\beta u_{,x}\,dx + \int_\Omega wf\,dx = 0 \quad \forall w \in \mathcal{V}$$

Want to integrate the first term by parts. Second term could be integrated by parts - but no clear advantage to gain - Leave it

$$-\int_\Omega w_{,x}\,\alpha u_{,x}\,dx + \underbrace{w\alpha u_{,x}\Big|_0^1}_{h} - \int_\Omega w\beta u_{,x}\,dx + \int_\Omega wf\,dx = 0 \quad \forall w \in \mathcal{V}$$

$$w\alpha u_{,x}\Big|_0^1 = w(1)\alpha u_{,x}(1) - \cancel{w(0)}\alpha u_{,x}(0)$$

(W) Given $f: \Omega \to \mathbb{R}$, $\alpha: \Omega \to \mathbb{R}$, $\beta: \Omega \to \mathbb{R}$ and constants g and h, Find $u: \Omega$, $u \in \mathcal{S}$ such that

$$\int_\Omega w_{,x}\alpha u_{,x}\,dx + \int_\Omega w\beta u_{,x}\,dx = \int_\Omega wf\,dx + w(1)h \quad \forall w \in \mathcal{V}$$

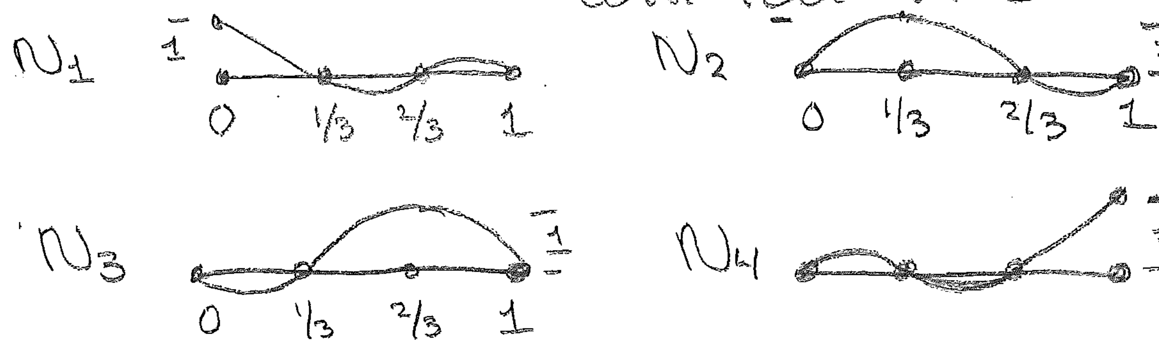
$$\mathcal{V} = \{w \mid w \in H^1, w(0) = 0\}$$

$$\mathcal{S} = \{u \mid u \in H^1, u(0) = g\}$$

For the discretization of this I want to use a

$$u^h = \sum_{A=1}^{4} N_a d_a$$

where my shape functions will look like -



What does each d_a represent?

$$u^h = v^h + g^h$$

What are the expressions for

$v^h = ?$

$g^h = ?$

Will the stiffness matrix for this problem be symmetric?

Is $\int_\Omega [(w_{,x} \alpha u_{,x}) + (w \beta u_{,x})] dx$ bi linear

Check bilinearity

$$a(w, c_1 u + c_2 v) = c_1 a(w,u) + c_2 a(w,v)$$

$$\int_\Omega w_{,x} b_1 (c_1 u + c_2 v)_{,x} dx + \int_\Omega w b_2 (c_1 u + c_2 v)_{,x} dx + \int_\Omega w b_3 (c_1 u + c_2 v) dx$$

note - $(c_1 u + c_2 v)_{,x} = c_1 u_{,x} + c_2 v_{,x}$

$$= c_1 \int_\Omega w_{,x} b_1 u_{,x} dx + c_2 \int_\Omega w_{,x} b_1 v_{,x} dx + c_1 \int_\Omega w b_2 u_{,x} dx + c_2 \int_\Omega w b_2 v_{,x} dx$$
$$+ c_1 \int_\Omega w b_3 u \, dx + c_2 \int_\Omega w b_3 v \, dx$$

$= c_1 a(w,u) + c_2 a(w,v)$ i.e. bilinear

$a(w,u) = a(u,w)$ look at middle term

$\int_\Omega w b_2 u_{,x} dx \neq \int_\Omega u b_2 w_{,x} dx$ Not Symmetric

OK - do not count on symmetry!

Comment on continuity between elements -

K - global stiffness matrix

k^e - stiffness matrix for element e

$$k^e = \int_{\Omega^e} \mathcal{D}^m(w^e; u^e)\, d\Omega^e$$ ← only over element (misses inter element boundaries)

$$K = \int_{\Omega} \sum_{e=1}^{nel} \mathcal{D}^m(w^e; u^e)\, d\Omega$$ Implies we are taking care of inter element boundaries

We want to do the individual k^e integrals - This requires pulling the Σ outside the integral

However, in general

$$K = \sum_{e=1}^{nel} \int_{\Omega^e} \mathcal{D}^m(w^e; u^e)\, d\Omega + \underbrace{\sum_{l=1}^{\#\text{ of inter-element boundaries}} \int_{\Gamma^l} \bar{\mathcal{D}}(w^e; u^e)\, d\Omega}$$

We want the second term to be zero

It turns out that so long as you have C^{m-1} order continuity between elements (which means only finite jumps in integrand between elements) we are set (inter element continuity)

Note - Need C^m continuity with element (intra element continuity)