Method of weighted residuals – procedure to construct weak forms from strong forms.

Given $f : \Omega \rightarrow \Re$ and known g_i, find $u : \bar{\Omega} \rightarrow \Re$ such that

$$D^{2m}(u) - f = 0 \text{ on } \Omega \quad \text{eq. A}$$

$$B^i(u) = g_i \text{ on } \Gamma_i \quad i = 0(1)(2m-1) \text{ eq. B}$$

D^j and B^j are jth order differential operators and Γ_i are appropriate portions of the boundary. Note that unlike the case of $m = 1$ where $\Gamma_g \cap \Gamma_h = 0$ and $\bar{\Gamma}_g \cup \bar{\Gamma}_h = \Gamma$ there can be more than one BC at a boundary location. For example for the $m = 2$ case there are two BC at every location on the boundary.

An example $m = 2$ case: Beam bending

Given $f : \Omega \rightarrow \Re$ and constants g_1, g_2, h_1, h_2, find $u : \bar{\Omega} \rightarrow \Re$ such that

$$EIu_{,xxxx} - f = 0 \text{ on } \Omega$$

$$u|_{\Gamma_u} = g_1 \text{ displacement BC}$$

$$u_{,x}|_{\Gamma_\Theta} = g_2 \text{ rotation BC}$$

$$EIu_{,xx}|_{\Gamma_M} = h_1 \text{ moment BC}$$

$$EIu_{,xxx}|_{\Gamma_Q} = h_2 \text{ shear BC}$$

Accepting the fact that we often not solve for the exact u, but can find an approximation to it, u^a meaning, we will not satisfy the differential equation (eq A) and/or the BC (eq B) fully. Thus in general we have:

$$D^{2m}(u^a) - f \neq 0$$

$$D^{2m}(u^a) - f = R, \ u^a \in \delta^a \subset \delta$$

The method of weighted residuals (MWR) seeks to minimize the residual, R, by forcing a weighted integral of the strong form domain equation (eq. A) to zero for an appropriate set of weighting functions

$$\int_\Omega w\left(D^{2m}(u^a) - f\right) d\Omega = 0 \ \ \forall w \in V$$

Since we cannot solve for $u^a \in \delta^a$ using an infinite dimension space V, we use a finite dimensional space

$$\int_\Omega w^a\left(D^{2m}(u^a) - f\right) d\Omega = 0 \ \ \forall w^a \in V^a \subset V$$

As always there will be a priori conditions that the members of the spaces V^a and δ^a must satisfy. What they are is a function of what integration by parts operations we apply. For our finite element methods we will typically want to take derivatives off of u, which means we put them on w. In the most common case we will integrate by parts m times so that highest order derivatives we see are m on both u and w.

There are a number of weighted residual methods:

Collocation – Force residual to zero at n specific points
$$\int_\Omega \delta(x - x_A)\left(D^{2m}(u^a) - f\right)d\Omega = 0$$
this yields
$$D^{2m}(u^a(x_A) - f) = 0, \quad A = 1(1)n$$

Least Squares – Want to minimize square of the residual integrated over the domain
$$Min.\left(\int_\Omega \left(D^{2m}\left(u(x,d_A)\right) - f\right)^2 d\Omega\right)$$
where d_A are unknown parameters. The min. satisfies
$$\frac{\partial}{\partial d_A}\left(\int_\Omega \left(D^{2m}\left(u(x,d_A)\right) - f\right)^2 d\Omega\right) = 0, \quad A = 1(1)n$$
this yields
$$\int_\Omega \frac{\partial\left(D^{2m}\left(u(x,d_A)\right) - f\right)}{\partial d_A}\left(D^{2m}\left(u(x,d_A)\right) - f\right)d\Omega = 0, \quad A = 1(1)n$$

Galerkin methods – alternative selections of w^a
As before we will decompose $u^a = v^a + g^a$.
If the shape functions, N_A, used for w^a and v^a are the same this is a Bubnov-Galerkin method, or just Galerkin.
If the shape functions, N_A, used for w^a and v^a are different this is a Petrov-Galerkin method.

Steps in applying a Galerkin approach

Step 1: Select weighting and trial functions in terms of shape functions, N_A:

$$u^h = v^h + g^h = d_A N_A + g_B N_B = \sum_{A=1}^{n} d_A N_A + \sum_{B=n+1}^{n+m} g_B N_B$$

$$w^h = c_A N_A = \sum_{A=1}^{n} c_A N_A$$

Step 2: Plug into the Galerkin criteria

$$c_A \int_\Omega N^A \left(D^{2m}(u^a) - f \right) d\Omega = 0 \quad A = 1(1)n$$

recalling the c_A are arbitrary, we have

$$\int_\Omega N^A \left(D^{2m}(u^a) - f \right) d\Omega = 0 \quad A = 1(1)n$$

Typically do not want to work from this form directly so we go to the next step.

Step 3: Apply integration by parts to the terms of interest until we get the form we like best. Symbolically:

$$\int_\Omega \bar{D}^k(N_A) D^l(u^h) d\Omega = \int_\Gamma \bar{D}^k(N_A) \breve{D}^{l-1}(u^h) d\Gamma - \int_\Omega \hat{D}^{k+1}(N_A) \hat{D}^{l-1}(u^h)$$

Step 4: Get the algebraic form

Lets look at an example of what we may do for constructing a stiffness matrix for a FE. Note that when doing a stand alone finite element, we will consider all possible unknowns (that is no essential boundary conditions, just natural boundary conditions). We will see the mechanics of how we assemble things will take of the boundary conditions.

Lets do an example (which is basically beam bending). A non-dimensional strong form is:

$$u_{,xxxx} + f = 0 \text{ in } 0 < x < 1$$

Subject to natural BC on both ends (for this case the essential BC are on u and $u_{,x}$)

$$u_{,xx}(0) = h_1 \text{ , } u_{,xx}(1) = h_2 \text{ , } u_{,xxx}(0) = h_3 \text{ , } u_{,xxx}(1) = h_4$$

Step 1: Select trial functions
This is actually a bit trickier than you may guess in that one needs to know basically what the final equations will look like to be sure to select trial functions that can satisfy the requirements. For a strong form with D^{2m} the best we can do in integrate m times to get the highest derivative on u and w to be m. In the current case $m = 2$, Thus $u \in H^2$ and $w \in H^2$. We will learn later that this will mean the shape functions will need to be high enough order to take second derivatives and, since we want to put multiple elements together in a mesh, will require that u and $u_{,x}$ are continuous between elements (interelement continuity requirement). I can meet the interelement continuity requirement for 1-D $m = 2$ problems by using "interpolating shape" functions that have $u(0)$ and $u_{,x}(0)$ and $u(1)$ and $u_{,x}(1)$ as unknowns. In 1-D this will require a polynomial with at least 4 parameters – A cubic polynomial has 4 parameters.

Thus we will use:

$$u^h = N_1 u(0) + N_2 u_{,x}(0) + N_3 u(1) + N_4 u_{,x}(1) = \sum_{a=1}^{4} N_a d_a$$

$$N_1 = 1 - 3x^2 + 2x^3$$

$$N_2 = x(x-1)^2$$

$$N_3 = 3x^2 - 2x^3$$

$$N_4 = x^3 - x^2$$
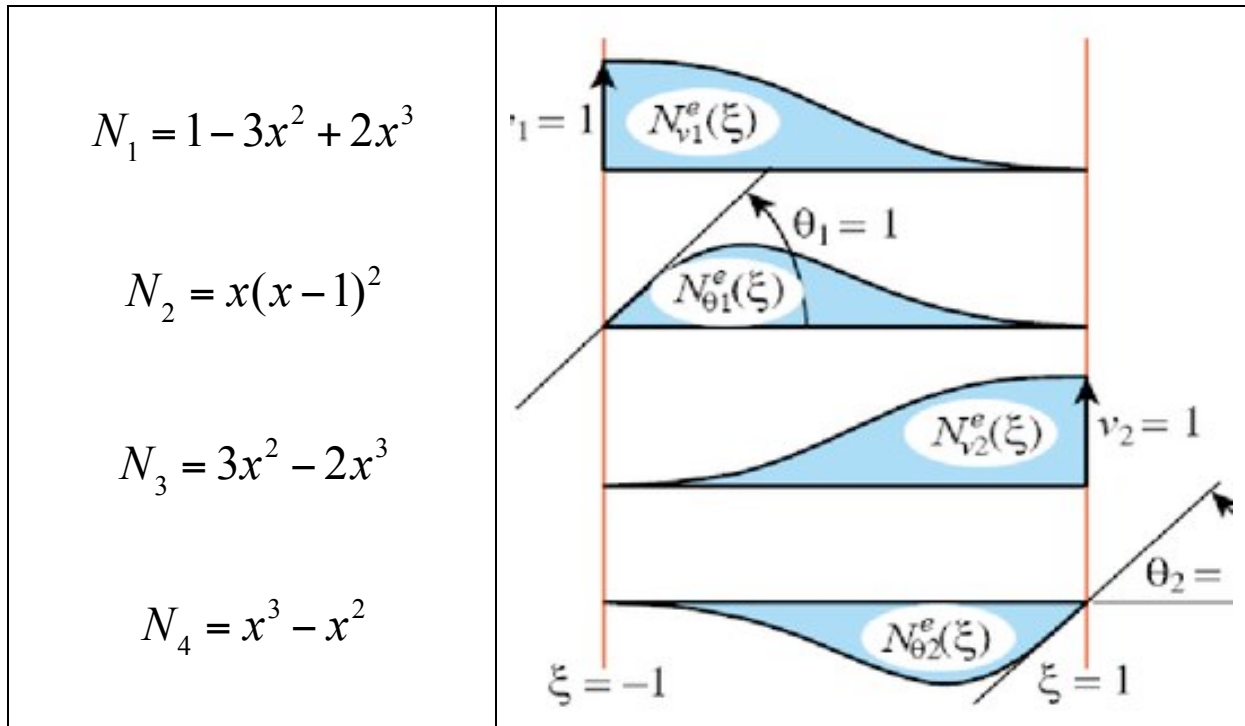
Note:

$$N_1(0) = 1, N_{1,x}(0) = 0, N_1(1) = 0, N_{1,x}(1) = 0$$
$$N_2(0) = 0, N_{2,x}(0) = 1, N_2(1) = 0, N_{2,x}(1) = 0$$
$$N_3(0) = 0, N_{3,x}(0) = 0, N_3(1) = 1, N_{3,x}(1) = 0$$
$$N_4(0) = 0, N_{4,x}(0) = 0, N_4(1) = 0, N_{4,x}(1) = 1$$

Step 2: Galerkin criteria

$$\int_0^1 N_a \left(u^h_{,xxxx} - f \right) dx = 0, \quad a = 1(1)4$$

note: changed from A to a since we will use the lowercase when we talk about elements

Step 3: Integrate by parts – look at the first term

$$\int_0^1 N_a u^h_{,xxxx} dx = N_a u^h_{,xxx} \big|_0^1 - \int_0^1 N_{a,x} u^h_{,xxx} dx$$

integrating again

$$\int_0^1 N_a u^h_{,xxxx}\,dx = N_a u^h_{,xxx}\Big|_0^1 - N_{a,x} u^h_{,xx}\Big|_0^1$$

$$+\int_0^1 N_{a,xx} u^h_{,xx}\,dx$$

Substituting this into Galerkin criteria we have the desired form:

$$\int_0^1 N_{a,xx} u^h_{,xx}\,dx = \int_0^1 N_a f\,dx - N_a u^h_{,xxx}\Big|_0^1$$

$$+N_{a,x} u^h_{,xx}\Big|_0^1, \quad a = 1(1)4$$

Step 4: Get Algebraic form
Note: Since we have no essential boundary conditions we will be constructing an element stiffness matrix.
For this we plug $u^h_{,xx}$, which is

$$u^h_{,xx} = \sum_{a=1}^4 N_{a,xx} d_a = N_{1,xx} d_1 + N_{2,xx} d_2 + N_{3,xx} d_3 + N_{4,xx} d_4$$

into the previous equation. This yields

$$\left[k^e\right]_{4x4} \{d^e\}_{4x1} = \{F^e\}_{4x1} + \{\bar{F}^e\}_{4x1}$$

where

$$k_{ab} = \int_0^1 N_{a,xx} N_{b,xx}\,dx \quad F_a^e = \int_0^1 N_a f\,dx \text{ and}$$

$$\bar{F}_a^e = N_{a,x} u^h_{,xx}\Big|_0^1 - N_a u^h_{,xxx}\Big|_0^1$$

Employing the given natural boundary conditions and the values of the shape functions and their derivatives at *0* and *1*, we have:

$$\bar{F}_1^e = h_3, \bar{F}_2^e = -h_1, \bar{F}_3^e = -h_4, \bar{F}_4^e = h_2$$