Will Cover §5.1, 5.2 and parts of 5.3 1

5

Plate and shell theories are derived by applying assumptions on 3D solid mechanics equations that yield reduced dimension equations

The C^0-Approach to Plates and Beams

Poisson-Kirchhoff plate theory applies normals remain straight and normal to strong form to yield 2D domain equation $\frac{d^4 w}{dx^4} + 2\frac{d^4 w}{dx^2 dy^2} + \frac{d^4 w}{dy^4} = q/D$

5.1 INTRODUCTION

C^1 in 2D a problem

The classical Poisson-Kirchhoff theory of plates requires C^1-continuity, just as does the classical Bernoulli-Euler beam theory (see Sec. 1.16). Continuous (i.e., C^0) finite element interpolations are easily constructed. The same cannot be said for multidimensional C^1-interpolations. It has taken considerable ingenuity to develop compatible C^1-interpolation schemes for two-dimensional plate elements based on classical theory, and the resulting schemes have always been extremely complicated in one way or another.

Almost standard today to not use Poisson-Kirchhoff

More and more, there is a turning away from Poisson-Kirchhoff type elements to elements based upon theories which accommodate transverse shear strains (Reissner and Mindlin theories) and require only C^0-continuity. This approach opens the way to a greater variety of interpolatory schemes but is not without its own inherent difficulties. Recently, displacement-type elements have been derived based upon Reissner–Mindlin theory, which seem to be superior to plate elements derived heretofore. This chapter discusses the basic techniques and considerations involved and summarizes recent developments in this area.

Following this, a similar approach is discussed in the context of beams and frames in which transverse shearing strains are accounted for. This also proves to be extremely simple and effective.

5.2 REISSNER–MINDLIN PLATE THEORY

5.2.1 Main Assumptions

All quantities are referred to a fixed system of rectangular, Cartesian coordinates. A

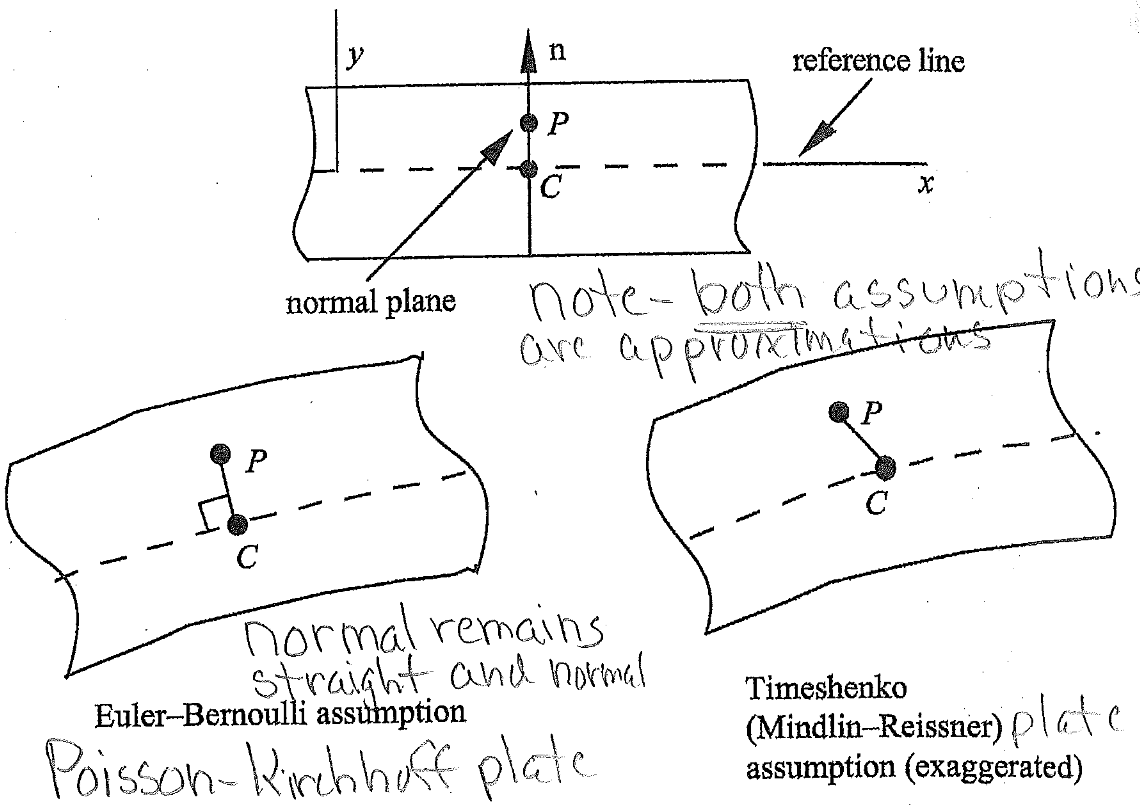


Figure 9.1 Motion in an Euler–Bernoulli beam and a shear (Timoshenko) beam; in the Euler–Bernoulli beam, the normal plane remains plane and normal, whereas in the shear beam the normal plane remains plane but not normal

properties as the linear strain since the equations for the rate-of-deformation can be obtained by replacing displacements by velocities in the linear strain–displacement relations. The aim of the following is to illustrate the consequences of the kinematic assumptions on the strain field, not to construct a theory which is worth implementing.

9.2.2 *Timoshenko (Shear Beam) Theory*

We first describe the Timoshenko beam theory. The major kinematic assumptions of this theory are that the normal planes remain plane, that is, flat, and that no deformation occurs within that plane. Thus the planes normal to the midline rotate as rigid bodies. Consider the motion of a point P whose orthogonal projection on the midline is point C. If the normal plane rotates as a rigid body, the velocity of point P relative to the velocity of point C is given by

$$\mathbf{v}_{PC} = \boldsymbol{\omega} \times \mathbf{r} \quad (9.2.1)$$

3

general point in this system is denoted by (x_1, x_2, x_3) or (x, y, z), whichever is more convenient. Throughout, Latin and Greek indices take on the values 1, 2, 3 and 1, 2, respectively.

x, y, z α, β

The main assumptions of the plate theory are

1. The domain Ω is of the following special form:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, z \in \left[\frac{-t}{2}, \frac{t}{2}\right], (x, y) \in A \subset \mathbb{R}^2\}$$

where t is the plate thickness and A is its area. The boundary of A is denoted by s.

2. $\sigma_{33} = 0$.
3. $u_\alpha(x, y, z) = -z\theta_\alpha(x, y)$.
4. $u_3(x, y, z) = w(x, y)$.

Remarks

1. In Assumption 1, we may take the plate thickness t to be a function of x and y, if desired. ← Get rid of z

major **2.** Assumption 2 is the plane stress hypothesis. It contradicts Assumption 4 but ultimately causes no problem. The justification of the present theory is its usefulness in practical structural engineering applications. No plate theory is completely consistent with the three-dimensional theory and, at the same time, both simple and useful. Assumption 2 is to be substituted into the constitutive equation; ϵ_{33} is to be solved for and subsequently eliminated.

3. Assumption 3 implies that plane sections remain plane. θ_α is interpreted as the rotation of a fiber initially normal to the plate midsurface (i.e., $z = 0$).

4. By Assumption 4, the transverse displacement, w, does not vary through the thickness.

The sign convention is illustrated in Fig. 5.2.1. "Right-hand-rule" rotations $\hat{\theta}_\alpha$ are defined by $\theta_\alpha = -e_{\alpha\beta}\hat{\theta}_\beta$, where $e_{\alpha\beta}$ is the alternator tensor, viz.,

Tracking the sign conventions is a real pain -

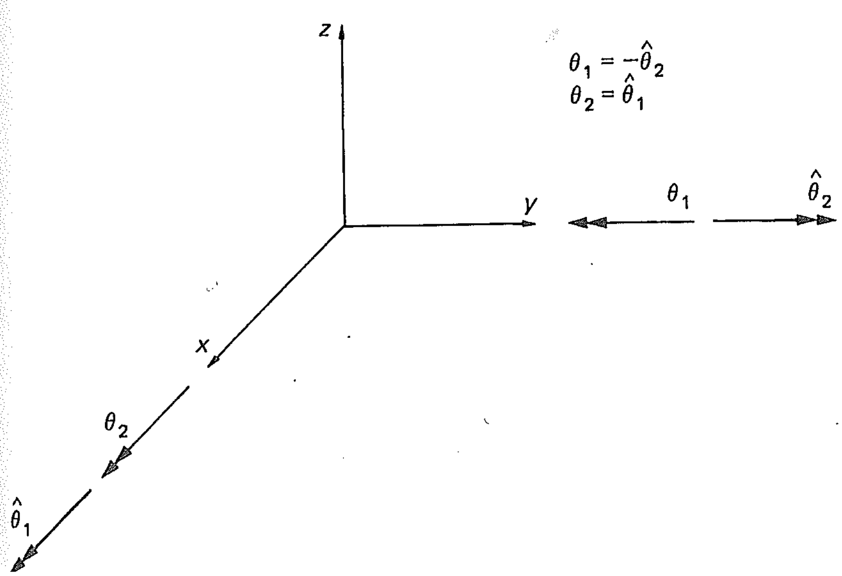


Figure 5.2.1 Sign conventions for rotations. $\hat{\theta}_1$, $\hat{\theta}_2$ are right-hand-rule rotations; θ_1, θ_2 are rotations that simplify the development of the plate theory.

4

$$[e_{\alpha\beta}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{5.2.1}$$

We prefer to develop the theory in terms of θ_α rather than $\hat{\theta}_\alpha$ because the algebra is greatly simplified, due to the absence of alternator tensors. In typical structural analysis computer programs, the right-hand-rule convention is usually, but not always, adopted. Consequently, it is the responsibility of the analyst to determine which convention is being employed. It is common when analysts use a new program that errors are made because of lack of careful attention to this point. *Whenever a plate or shell analysis is being undertaken the analyst should check the rotation–bending moment sign convention of the computer program being used before embarking upon that analysis.*

Plate kinematics are summarized in Fig. 5.2.2.

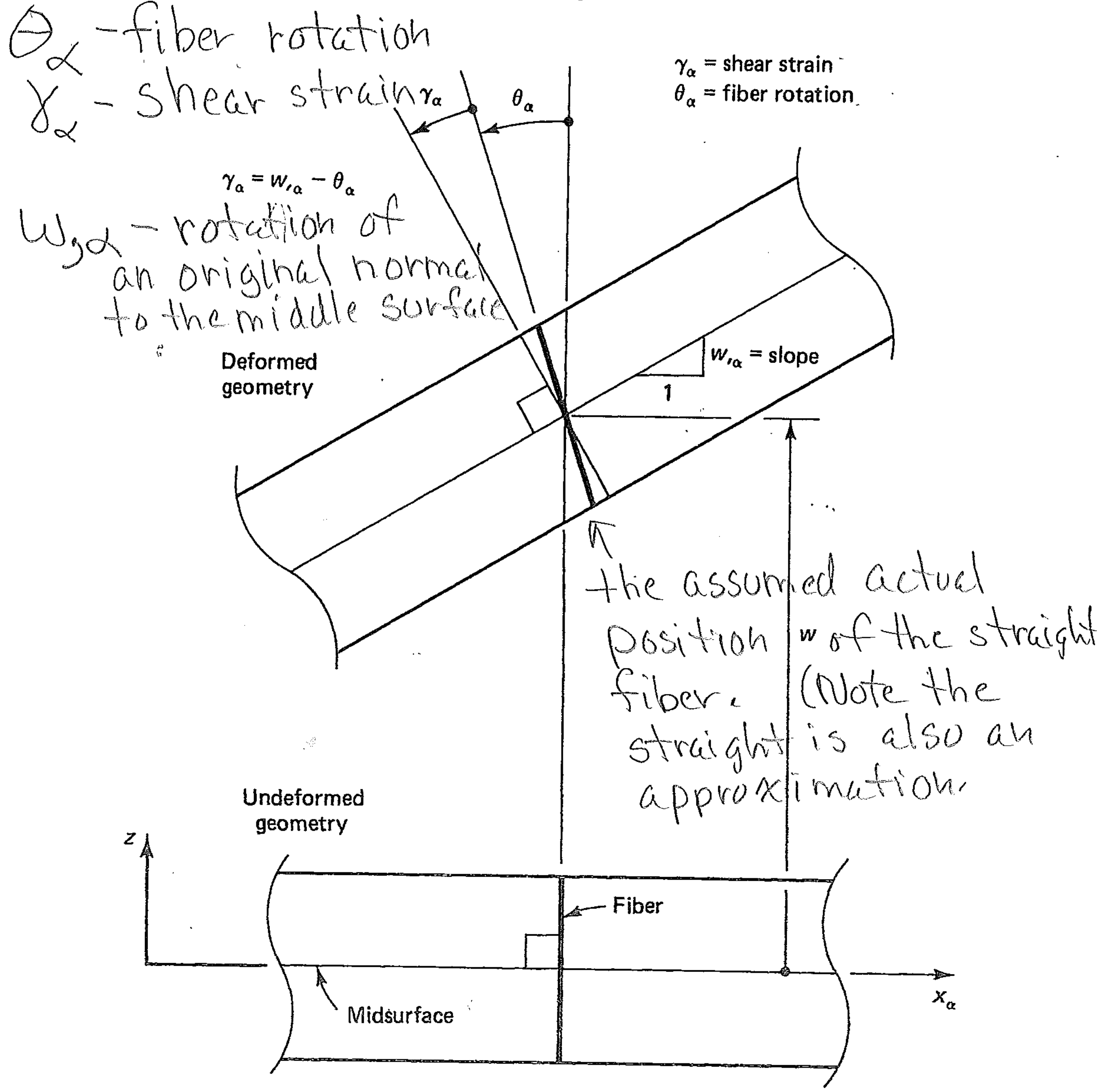


Figure 5.2.2 Plate kinematics. Transverse shear strains need not vanish in the present theory.

5

5.2.2 Constitutive Equation

The reduced form of the constitutive equation used in the plate theory is determined by substituting Assumption 2 into the three-dimensional constitutive equation and eliminating ϵ_{33}. For simplicity, we shall consider the isotropic case in which

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \tag{5.2.2}$$

(for 3D linear elastic

where λ and μ are the Lamé coefficients and δ_{ij} is the Kronecker delta. Assumption 2 implies $\sigma_{33} = \lambda(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2\mu\epsilon_{33} = 0$

solving for ϵ_{33}

$$\epsilon_{33} = \frac{-\lambda}{\lambda + 2\mu} \epsilon_{\alpha\alpha} \tag{5.2.3}$$

k $(\epsilon_{11}+\epsilon_{22})$

after a tad of manipulation we have

$$\boxed{\sigma_{\alpha\beta} = \bar{\lambda} \delta_{\alpha\beta} \epsilon_{\gamma\gamma} + 2\mu \epsilon_{\alpha\beta}} \tag{5.2.4}$$

$$\boxed{\sigma_{\alpha 3} = 2\mu \epsilon_{\alpha 3}} \tag{5.2.5}$$

α & β go 1, 2

where

$$\bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu} \tag{5.2.6}$$

$\bar{\lambda}$ and μ may be eliminated in favor of E and ν (Young's modulus and Poisson's ratio, respectively):

$\epsilon_{\alpha\beta} = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2}$

$u_\alpha = -z\theta_\alpha$ — assumption 3

$u_{\alpha,\beta} = -z\theta_{\alpha,\beta}$

$$\bar{\lambda} = \frac{\nu E}{1 - \nu^2} \tag{5.2.7}$$

$$\mu = \frac{E}{2(1 + \nu)} \tag{5.2.8}$$

$\epsilon_{\alpha,3} = \frac{u_{\alpha,3} + u_{3,\alpha}}{2}$

$w \equiv u_3$ ← assumption 4

$u_\alpha = -z\theta_\alpha$

$u_{\alpha,z} = -\theta_\alpha$

5.2.3 Strain-displacement Equations

Assumptions 3 and 4 lead to the following form of the strain-displacement equations:

$$\epsilon_{\alpha\beta} = u_{(\alpha,\beta)} = -z\theta_{(\alpha,\beta)} \tag{5.2.9}$$

$$\epsilon_{\alpha 3} = u_{(\alpha,3)} = \frac{-\theta_\alpha + w_{,\alpha}}{2} \tag{5.2.10}$$

Note that the normal-fiber rotation (i.e., θ_α) and slope (i.e., $w_{,\alpha}$) are not necessarily the same and thus transverse shear strains are accommodated. This is to be contrasted with *classical Poisson-Kirchhoff (i.e., "thin plate") theory* in which $\theta_\alpha = w_{,\alpha}$ and, consequently, $\epsilon_{\alpha 3} = 0$. In the thin plate limit, we usually expect very small transverse shear strains.

5.2.4 Summary of Plate Theory Notations

w	(transverse displacement)
θ_α	(rotation vector)
$\kappa_{\alpha\beta} = \theta_{(\alpha,\beta)}$	(curvature tensor)
$\gamma_\alpha = -\theta_\alpha + w_{,\alpha}$	(shear strain vector)
$m_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta} z \, dz$	(moment tensor)
$q_\alpha = \int_{-t/2}^{t/2} \sigma_{\alpha 3} \, dz$	(shear force vector)
W	(prescribed boundary displacement)
Θ_α	(prescribed boundary rotations)
$F = \int_{-t/2}^{t/2} \ell_3 dz + \langle h_3 \rangle$[1]	(total applied transverse force per unit area)
$C_\alpha = \int_{-t/2}^{t/2} \ell_\alpha z \, dz + \langle h_\alpha z \rangle$	(total applied couple per unit area)
$M_\alpha = \int_{-t/2}^{t/2} h_\alpha z \, dz$	(prescribed boundary moments)
$Q = \int_{-t/2}^{t/2} h_3 \, dz$	(prescribed boundary shear force)

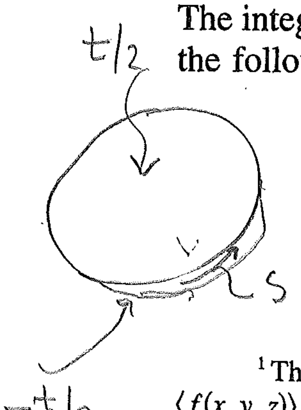
Terms to be introduced as we go

5.2.5 Variational Equation

Instead of applying the assumptions to the strong form - Will apply to integral form and integrate the z octo

The variational equation of the plate theory is derived from the variational equation of the three-dimensional theory by making use of the preceding relations. The main steps are as follows.

i. Let s_q and s_h be subregions of s which satisfy $\overline{s_q \cup s_h} = s$ and $s_q \cap s_h = \varnothing$. The integrals appearing in the three-dimensional variational equations are replaced by the following iterated integrals:



$$\int_\Omega \ldots d\Omega = \int_A \int_{-t/2}^{t/2} \ldots dz \, dA \tag{5.2.11}$$

around the plate

$$\int_{\Gamma_h} \ldots d\Gamma = \int_A \langle \ldots \rangle \, dA + \int_{s_h} \int_{-t/2}^{t/2} \ldots dz \, ds \tag{5.2.12}$$

Top & bottom & Thickness

[1] The operator $\langle \; \rangle$ is defined as follows: Let f be an arbitrary function of x, y, and z. Then $\langle f(x,y,z) \rangle = f(x,y,-t/2) + f(x,y,t/2)$.

Start with variational (Weak) form of equations of 3D elasticity

$$0 = \underbrace{\int_\Omega \bar{u}_{(i,j)} \sigma_{ij} \, d\Omega}_{\text{term } 1} - \underbrace{\int_\Omega \bar{u}_i f_i \, d\Omega}_{\text{term } 2} - \underbrace{\int_\Gamma \bar{u}_i h_i \, d\Gamma}_{\text{term } 3}$$

$\bar{u}_{(i,j)}, \bar{u}_i \leftarrow$ weighting terms

Using equations 5.2.11 and 5.2.12 to decompose the domain and boundary integrals, and substituting the terms we have derived based on the R/M assumptions we have

Term 1

$$\int_\Omega \bar{u}_{(i,j)} \sigma_{ij} \, d\Omega = \int_A \int_{-t/2}^{t/2} \left[\bar{u}_{(\alpha,\beta)} \sigma_{\alpha,\beta} + 2 \bar{u}_{(\alpha,3)} \sigma_{\alpha,3} \right] dz \, dA$$

using $u_{(\alpha,\beta)} = -z\, \theta_{(\alpha,\beta)} \equiv -z\, \underset{\substack{\uparrow \\ \text{curvature}}}{\kappa}_{\alpha\beta}$

$$u_{(\alpha,3)} = \frac{-\theta_\alpha + w_{,\alpha}}{2} \equiv \underset{\substack{\uparrow \\ \text{shear strain}}}{\gamma_\alpha}/2$$

we get

$$\int_A \int_{-t/2}^{t/2} \left[-\bar{\kappa}_{\alpha\beta} \sigma_{\alpha\beta} z + \bar{\gamma}_\alpha \sigma_{\alpha 3} \right] dz \, dA$$

Term (2)

$$\int_\Omega \bar{u}_i f_i \, d\Omega = \int_A \int_{-t/2}^{t/2} \bar{u}_\alpha f_\alpha + \bar{u}_3 f_3 \, dz \, dA$$

Using $u_\alpha = \theta_\alpha z$ and $u_3 = w$

↑ rotation ↑ transvers disp.

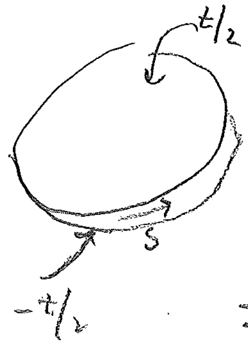
$$\int_A \int_{t/2}^{t/2} [-\bar{\theta}_\alpha f_\alpha z + \bar{w} f_3) \, dz \, dA$$

Term (3)

top and bottom

$$\int_{\Gamma_h} \bar{u}_i h_i = \int_A [\langle \bar{u}_\alpha h_\alpha + \bar{u}_3 h_3 \rangle dA + \int_{S_h} \int_{-t/2}^{t/2} [\bar{u}_\alpha h_\alpha + \bar{u}_3 h_3) \, dz \, ds$$

side



Using $u_\alpha = \theta_\alpha z, \; u_3 = w$

$$= \int_A [-\bar{\theta}_\alpha \langle h_\alpha z \rangle + \bar{w} \langle h_3 \rangle dA + \int_{S_h} \int_{-t/2}^{t/2} [-\bar{\theta}_\alpha h_\alpha z + \bar{w} h_3] \, dz \, ds$$

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The kinematic relations are also employed, yielding

Collecting things back together

$$
\begin{aligned}
0 = & \int_A \int_{-t/2}^{t/2} [\overline{u}_{(\alpha,\beta)}\sigma_{\alpha\beta} + 2\overline{u}_{(\alpha,3)}\sigma_{\alpha 3}]\, dz\, dA \\
& - \int_A \int_{-t/2}^{t/2} (\overline{u}_\alpha \ell_\alpha + \overline{u}_3 \ell_3)\, dz\, dA \\
& - \int_A (\langle \overline{u}_\alpha h_\alpha \rangle + \langle \overline{u}_3 h_3 \rangle)\, dA \\
& - \int_{s_h} \int_{-t/2}^{t/2} (\overline{u}_\alpha h_\alpha + \overline{u}_3 h_3)\, dz\, ds
\end{aligned}
$$

before substitutions

$$
\begin{aligned}
= & \int_A \int_{-t/2}^{t/2} (-\overline{\kappa}_{\alpha\beta}\sigma_{\alpha\beta} z + \overline{\gamma}_\alpha \sigma_{\alpha 3})\, dz\, dA \\
& - \int_A \int_{-t/2}^{t/2} (-\overline{\theta}_\alpha \ell_\alpha z + \overline{w}\ell_3)\, dz\, dA \\
& - \int_A (-\overline{\theta}_\alpha \langle h_\alpha z \rangle + \overline{w}\langle h_3 \rangle)\, dA \\
& - \int_{s_h} \int_{-t/2}^{t/2} (-\overline{\theta}_\alpha h_\alpha z + \overline{w} h_3)\, dz\, ds
\end{aligned}
\tag{5.2.13}
$$

after substitutions

where

$$\overline{\kappa}_{\alpha\beta} = \overline{\theta}_{(\alpha,\beta)} \tag{5.2.14}$$

$$\overline{\gamma}_\alpha = -\overline{\theta}_\alpha + \overline{w}_{,\alpha} \tag{5.2.15}$$

Note. In the preceding relations we have used quantities with superposed bars to denote weighting functions in order to avoid notational conflicts and a proliferation of new notations.

ii. The definitions of force resultants are used, yielding

$$
\boxed{
\begin{aligned}
0 = & \int_A (-\overline{\kappa}_{\alpha\beta} m_{\alpha\beta} + \overline{\gamma}_\alpha q_\alpha)\, dA \\
& - \int_A (-\overline{\theta}_\alpha C_\alpha + \overline{w} F)\, dA \\
& - \int_{s_h} (-\overline{\theta}_\alpha M_\alpha + \overline{w} Q)\, ds
\end{aligned}
}
\tag{5.2.16}
$$

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iii. Integration by parts indicates, under the usual hypotheses, the differential equations and boundary conditions that are satisfied:

Looking at the Euler–Lagrange equations – seeing ingredients of the strong form

$$0 = \int_A \bar{\theta}_\alpha \underbrace{(m_{\alpha\beta,\beta} - q_\alpha + C_\alpha)}_{\text{moment equilibrium}} dA$$

$$- \int_A \bar{w} \underbrace{(q_{\alpha,\alpha} + F)}_{\substack{\text{transverse} \\ \text{equilibrium}}} dA$$

$$+ \int_{s_h} \{\bar{\theta}_\alpha \underbrace{(-m_{\alpha n} + M_\alpha)}_{\substack{\text{moment} \\ \text{boundary} \\ \text{conditions}}} + \bar{w} \underbrace{(q_n - Q)}_{\substack{\text{shear} \\ \text{boundary} \\ \text{condition}}}\}\, ds \tag{5.2.17}$$

where

$$m_{\alpha n} = m_{\alpha\beta} n_\beta \tag{5.2.18}$$

$$q_n = q_\alpha n_\alpha \tag{5.2.19}$$

iv. Explicit forms of the constitutive equations in terms of the plate-theory variables are computed as follows:

We want to integrate out all the through the thickness integrals

$$\begin{aligned} m_{\alpha\beta} &= \int_{-t/2}^{t/2} \sigma_{\alpha\beta} z \, dz \quad \leftarrow \text{moment tensor} \\ &= \int_{-t/2}^{t/2} (\bar{\lambda} \delta_{\alpha\beta} \epsilon_{\gamma\gamma} + 2\mu \epsilon_{\alpha\beta}) z \, dz \\ &= -\frac{t^3}{12} [\bar{\lambda} \delta_{\alpha\beta} \theta_{\gamma,\gamma} + 2\mu \theta_{(\alpha,\beta)}] \\ &= -c_{\alpha\beta\gamma\delta} \kappa_{\gamma\delta} \end{aligned} \tag{5.2.20}$$

where

isotropic $\rightarrow$

$$c_{\alpha\beta\gamma\delta} = \frac{t^3}{12} [\mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + \bar{\lambda} \delta_{\alpha\beta} \delta_{\gamma\delta}] \tag{5.2.21}$$

$$\begin{aligned} q_\alpha &= \int_{-t/2}^{t/2} \sigma_{\alpha 3} \, dz \quad \leftarrow \text{shear force} \\ &= \int_{-t/2}^{t/2} 2\mu \epsilon_{\alpha 3} \, dz \\ &= t\mu(-\theta_\alpha + w_{,\alpha}) \\ &= c_{\alpha\beta} \gamma_\beta \end{aligned} \tag{5.2.22}$$

$$c_{\alpha\beta} = t\mu \delta_{\alpha\beta} \tag{5.2.23}$$

$$F = \int_{-t/2}^{t/2} f_3 \, dz + \langle h_3 \rangle$$ transverse force per unit area

$$C_\alpha = \int_{-t/2}^{t/2} f_\alpha z \, dz + \langle h_\alpha z \rangle$$ Applied couple per unit area

$$m_\alpha = \int_{-t/2}^{t/2} h_\alpha z \, dz$$ prescribed boundary moment (line moment)

$$Q = \int_{-t/2}^{t/2} h_3 \, dz$$ prescribed boundary shear - line shear

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Remarks

1. Symmetry of the stiffness matrix will follow from the symmetries

$$c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta} \tag{5.2.24}$$

$$c_{\alpha\beta} = c_{\beta\alpha} \tag{5.2.25}$$

The additional symmetries

$$c_{\alpha\beta\gamma\delta} = c_{\beta\alpha\gamma\delta} = c_{\alpha\beta\delta\gamma} \tag{5.2.26}$$

also hold.

2. To achieve results consistent with classical bending theory it is necessary to introduce a shear correction factor, κ, in the shear force–shear strain constitutive equation. This can be done by replacing $c_{\alpha\beta}$ by $\kappa c_{\alpha\beta}$. Throughout it is assumed that $\kappa = \frac{5}{6}$. ← See pg 13

3. More-general material behavior (e.g., orthotropy) can be considered by appropriately redefining the elastic coefficients $c_{\alpha\beta\gamma\delta}$ and $c_{\alpha\beta}$.

5.2.6 Strong Form

The formal statement of the strong form of the plate theory boundary-value problem is as follows.

Given F, C_α, M_α, Q, W, and Θ_α, find w and θ_α such that

The result of the Euler-Lagrange process yields this strong form

$$\left.\begin{aligned} m_{\alpha\beta,\beta} - q_\alpha + C_\alpha &= 0 && (5.2.27)\\ q_{\alpha,\alpha} + F &= 0 && (5.2.28)\\ m_{\alpha\beta} &= -c_{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} && (5.2.29)\\ q_\alpha &= c_{\alpha\beta}\gamma_\beta && (5.2.30)\\ \kappa_{\alpha\beta} &= \theta_{(\alpha,\beta)} && (5.2.31)\\ \gamma_\alpha &= -\theta_\alpha + w_{,\alpha} && (5.2.32) \end{aligned}\right\} \text{ in } A$$

$$\left.\begin{aligned} \theta_\alpha &= \Theta_\alpha && (5.2.33)\\ w &= W && (5.2.34) \end{aligned}\right\} \text{ on } s_g$$

$$\left.\begin{aligned} m_{\alpha n} = m_{\alpha\beta}n_\beta &= M_\alpha && (5.2.35)\\ q_n = q_\alpha n_\alpha &= Q && (5.2.36) \end{aligned}\right\} \text{ on } s_h$$

Sign conventions for stress resultants are depicted in Fig. 5.2.3.

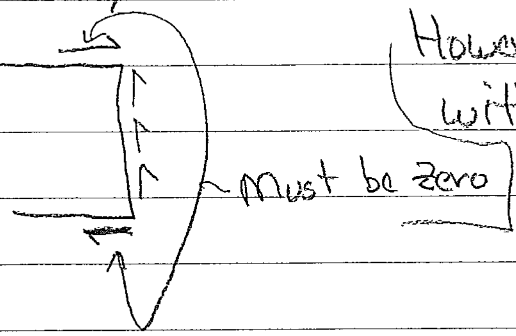
5.2.7 Weak Form

The statement of the variational, or weak, form of the boundary-value problem is as follows.

Given F, C_α, M_α, Q, W, and Θ_α, find $\{\theta_1, \theta_2, w\} \in \mathcal{S}$ such that, for all $\{\overline{\theta}_1, \overline{\theta}_2, \overline{w}\} \in \mathcal{V}$

Shear Correction –

Our Kinematic assumption ~~is~~ is uniform Shear strains through the thickness ⇐ normal remained Straight. However, this would correspond to having shear stresses at top and bottom –



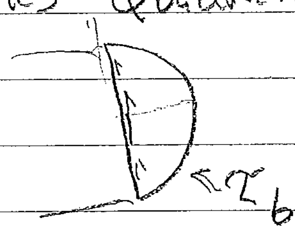
However, this is inconsistent with other assumptions and differential moment equilibrium – they need to be zero at top and bottom.

If we look back at basic beam / plate bending (thin) the shear stress consistent with the limit is

$$\tau_b = \frac{-VQ}{IT}$$

V – shear force
Q – "moment" of area above where we are
I – moment of inertia
T – thickness

Q varies Quadratically through thickness



by equating the "energy" contribution of this to our case with a correction k introduced in the "constitutive" term $C_{\alpha\beta}$ we find

$K = 5/6$ ⇐ note as plates get "thick" ~~this~~ it may be desirable to modify this a bit.

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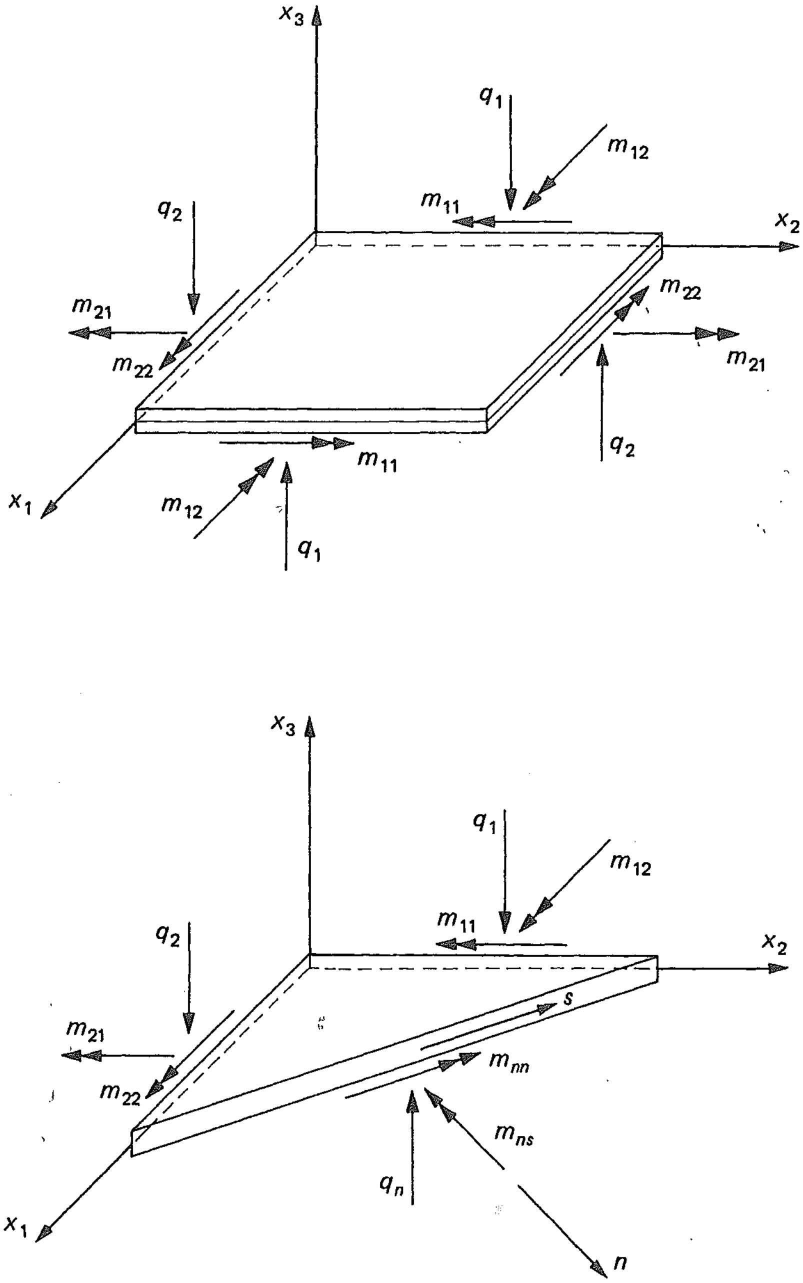

Figure 5.2.3 Sign convention for stress resultants.

weak form

$$\boxed{\begin{aligned} 0 = &\int_A [\bar{\theta}_{(\alpha,\beta)} c_{\alpha\beta\gamma\delta}\, \theta_{(\gamma,\delta)} + \bar{\gamma}_\alpha c_{\alpha\beta} \gamma_\beta]\, dA \\ &+ \int_A (\bar{\theta}_\alpha C_\alpha - \bar{w} F)\, dA \\ &+ \int_{S_h} (\bar{\theta}_\alpha M_\alpha - \bar{w} Q)\, ds \end{aligned}} \tag{5.2.37}$$

We assume that if

$$\boldsymbol{u} = \begin{Bmatrix} w \\ \theta_1 \\ \theta_2 \end{Bmatrix} \in \mathcal{S} \qquad \text{(the trial solution space)}$$

then

$$\begin{Bmatrix} w \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} W \\ \Theta_1 \\ \Theta_2 \end{Bmatrix} \qquad \text{on } s_g$$

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and that if

$$u = \begin{Bmatrix} \overline{w} \\ \overline{\theta}_1 \\ \overline{\theta}_2 \end{Bmatrix} \in \mathcal{V} \qquad \text{(the weighting function space)}$$

then

$$\begin{Bmatrix} \overline{w} \\ \overline{\theta}_1 \\ \overline{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{on } s_g$$

Exercise 1. Show that (5.2.27), (5.2.28), (5.2.35), and (5.2.36) are implied by (5.2.37).

Exercise 2. Put (5.2.37) into abstract notation: $a(\overline{u}, u) = (\overline{u}, \ell) + (\overline{u}, h)_\Gamma$. Define ℓ and h and show that $a(\cdot, \cdot)$; $(\cdot, \cdot)$; and $(\cdot, \cdot)_\Gamma$ are symmetric, bilinear forms.

5.2.8 Matrix Formulation

The matrix formulation of the variational equation is given as follows.

$$\boxed{\begin{aligned} 0 = &\int_A (\overline{\boldsymbol{\kappa}}^T D^b \boldsymbol{\kappa} + \overline{\boldsymbol{\gamma}}^T D^s \boldsymbol{\gamma})\, dA \\ &+ \int_A (\overline{\boldsymbol{\theta}}^T C - \overline{w} F)\, dA \\ &+ \int_{s_h} (\overline{\boldsymbol{\theta}}^T M - \overline{w} Q)\, ds \end{aligned}} \tag{5.2.38}$$

where

$$\boldsymbol{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \qquad \overline{\boldsymbol{\theta}} = \begin{Bmatrix} \overline{\theta}_1 \\ \overline{\theta}_2 \end{Bmatrix} \tag{5.2.39}$$

$$\boldsymbol{\gamma} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} \qquad \overline{\boldsymbol{\gamma}} = \begin{Bmatrix} \overline{\gamma}_1 \\ \overline{\gamma}_2 \end{Bmatrix} \tag{5.2.40}$$

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix} \qquad \overline{\boldsymbol{\kappa}} = \begin{Bmatrix} \overline{\kappa}_{11} \\ \overline{\kappa}_{22} \\ 2\overline{\kappa}_{12} \end{Bmatrix} \tag{5.2.41}$$

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$$D^b = \begin{bmatrix} D^b_{11} & D^b_{12} & D^b_{13} \\ & D^b_{22} & D^b_{23} \\ \text{symmetric} & & D^b_{33} \end{bmatrix} \tag{5.2.42}$$

$$D^b_{IJ} = c_{\alpha\beta\gamma\delta}, \tag{5.2.43}$$

I/J	α/γ	β/δ
1	1	1
2	2	2
3	1	2

$$D^s = \begin{bmatrix} D^s_{11} & D^s_{12} \\ \text{symm.} & D^s_{22} \end{bmatrix} \tag{5.2.44}$$

$$D^s_{\alpha\beta} = c_{\alpha\beta} \tag{5.2.45}$$

5.2.9 Finite Element Stiffness Matrix and Load Vector

The finite element stiffness matrix and load vector may be obtained directly from the matrix form of the variational equation. The finite element approximations of w, $\overline{w}$, θ_α, and $\overline{\theta}_\alpha$ are denoted by w^h, $\overline{w}^h$, θ^h_α, and $\overline{\theta}^h_\alpha$, respectively. In a typical element, possessing n_{en} nodes,

$$w^h = \sum_{a=1}^{n_{en}} N_a w^h_a \tag{5.2.46}$$

$$\overline{w}^h = \sum_{a=1}^{n_{en}} N_a \overline{w}^h_a \tag{5.2.47}$$

$$\theta^h_\alpha = \sum_{a=1}^{n_{en}} N_a \theta^h_{\alpha a} \tag{5.2.48}$$

$$\overline{\theta}^h_\alpha = \sum_{a=1}^{n_{en}} N_a \overline{\theta}^h_{\alpha a} \tag{5.2.49}$$

where N_a is the shape function associated with node a, and w^h_a, $\overline{w}^h_a$, $\theta^h_{\alpha a}$, and $\overline{\theta}^h_{\alpha a}$ are the ath nodal values of w^h, $\overline{w}^h$, θ^h_α, and $\overline{\theta}^h_\alpha$, respectively.

Remark

It is not necessary to assume θ^h_α and w^h are defined in terms of the same shape functions and nodal patterns. However, in the applications we have in mind, this will be the case.

Define

$$d^e = \{d^e_p\} \tag{5.2.50}$$

$$\bar{\boldsymbol{d}}^e = \{\bar{d}_p^e\} \tag{5.2.51}$$

$$d_p^e = \begin{cases} w_a^h & p = 3a - 2 \\ \theta_{1a}^h & p = 3a - 1 \\ \theta_{2a}^h & p = 3a \end{cases} \tag{5.2.52}$$

$$\bar{d}_p^e = \begin{cases} \bar{w}_a^h & p = 3a - 2 \\ \bar{\theta}_{1a}^h & p = 3a - 1 \\ \bar{\theta}_{2a}^h & p = 3a \end{cases} \tag{5.2.53}$$

$$\boldsymbol{\kappa} = \boldsymbol{B}^b \boldsymbol{d}^e \qquad \bar{\boldsymbol{\kappa}} = \boldsymbol{B}^b \bar{\boldsymbol{d}}^e \tag{5.2.54}$$

$$\boldsymbol{\gamma} = \boldsymbol{B}^s \boldsymbol{d}^e \qquad \bar{\boldsymbol{\gamma}} = \boldsymbol{B}^s \bar{\boldsymbol{d}}^e \tag{5.2.55}$$

$$\boldsymbol{B}^b = [\boldsymbol{B}_1^b, \boldsymbol{B}_2^b, \ldots, \boldsymbol{B}_{n_{en}}^b] \tag{5.2.56}$$

$$\boldsymbol{B}^s = [\boldsymbol{B}_1^s, \boldsymbol{B}_2^s, \ldots, \boldsymbol{B}_{n_{en}}^s] \tag{5.2.57}$$

$$\boldsymbol{B}_a^b = \begin{bmatrix} 0 & N_{a,x} & 0 \\ 0 & 0 & N_{a,y} \\ 0 & N_{a,y} & N_{a,x} \end{bmatrix} \tag{5.2.58}$$

$$\boldsymbol{B}_a^s = \begin{bmatrix} N_{a,x} & -N_a & 0 \\ N_{a,y} & 0 & -N_a \end{bmatrix} \tag{5.2.59}$$

With these definitions, the following expressions for the element stiffness and load may be obtained:

$$\boldsymbol{k}^e = \boldsymbol{k}_b^e + \boldsymbol{k}_s^e \tag{5.2.60}$$

$$\boldsymbol{k}_b^e = \int_{A^e} \boldsymbol{B}^{bT} \boldsymbol{D}^b \boldsymbol{B}^b \, dA \qquad \text{(bending stiffness)} \tag{5.2.61}$$

$$\boldsymbol{k}_s^e = \int_{A^e} \boldsymbol{B}^{sT} \boldsymbol{D}^s \boldsymbol{B}^s \, dA \qquad \text{(shear stiffness)} \tag{5.2.62}$$

$$\boldsymbol{f}^e = \{f_p^e\} \tag{5.2.63}$$

$$f_p^e = \begin{cases} \displaystyle\int_{A^e} N_a F \, dA + \int_{s^e \cap s_h} N_a Q \, ds & p = 3a - 2 \\ \displaystyle -\int_{A^e} N_a C_1 \, dA - \int_{s^e \cap s_h} N_a M_1 \, ds & p = 3a - 1 \\ \displaystyle -\int_{A^e} N_a C_2 \, dA - \int_{s^e \cap s_h} N_a M_2 \, ds & p = 3a \end{cases} \tag{5.2.64}$$

A^e and s^e are the area and the boundary, respectively, of the eth element. The adjustment to f_p^e for prescribed displacements is given by

$$f_p^e \leftarrow f_p^e - \sum_{q=1}^{n_{ee}} k_{pq}^e g_q, \qquad n_{ee} = 3n_{en} \tag{5.2.65}$$

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where

$$g_p = \begin{cases} W(x_a, y_a) & p = 3a - 2 \\ \Theta_1(x_a, y_a) & p = 3a - 1 \\ \Theta_2(x_a, y_a) & p = 3a \end{cases} \tag{5.2.66}$$

The element stresses may be obtained from the following relations:

$$\begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} = -\boldsymbol{D}^b \boldsymbol{B}^b \boldsymbol{d}^e \qquad \text{(bending moments)} \tag{5.2.67}$$

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \boldsymbol{D}^s \boldsymbol{B}^s \boldsymbol{d}^e \qquad \text{(shear resultants)} \tag{5.2.68}$$

Exercise 3. (Arrays with Respect to Right-hand-rule Rotations.) Show that if right-hand-rule rotations are being employed, i.e., if $\theta_{\alpha a}^h \leftarrow \hat{\theta}_{\alpha a}^h$ in (5.2.48), and, likewise, if $\overline{\theta}_{\alpha a}^h \leftarrow \hat{\overline{\theta}}_{\alpha a}^h$ in (5.2.49), then in place of (5.2.58), (5.2.59), and (5.2.64), respectively, we need to use

$$\boldsymbol{B}_a^b = \begin{bmatrix} 0 & 0 & -N_{a,x} \\ 0 & N_{a,y} & 0 \\ 0 & N_{a,x} & -N_{a,y} \end{bmatrix}$$

$$\boldsymbol{B}_a^s = \begin{bmatrix} N_{a,x} & 0 & N_a \\ N_{a,y} & -N_a & 0 \end{bmatrix}$$

$$f_p^e = \begin{cases} \text{same as in (5.2.64)}, & p = 3a - 2 \\ -\int_{A^e} N_a C_2 \, dA - \int_{s^e \cap s_h} N_a M_2 \, ds & p = 3a - 1 \\ \int_{A^e} N_a C_1 \, dA + \int_{s^e \cap s_h} N_a M_1 \, ds & p = 3a \end{cases}$$

5.3 PLATE-BENDING ELEMENTS

5.3.1 Some Convergence Criteria

It is important to realize that convergence criteria for elements derived from the present theory are quite different than those for elements derived from thin plate theory. *Necessary* conditions in the present case are:

1. All three rigid body modes must be exactly representable
2. The following five constant strain states must be exactly representable:

$$\left.\begin{array}{l}\theta_{1,1}\\ \theta_{2,2}\\ \frac{1}{2}(\theta_{1,2}+\theta_{2,1})\end{array}\right\}\quad\text{(curvatures)}$$

$$\left.\begin{array}{l}-\theta_1 + w_{,1}\\ -\theta_2 + w_{,2}\end{array}\right\}\quad\text{(transverse shear strains)}$$

These conditions are satisfied for standard isoparametric elements and for the non-standard isoparametric elements described in the following sections.

5.3.2 Shear Constraints and Locking

An important consideration in the development of plate-bending elements, based upon the present theory, is the number of shear strain constraints engendered in the thin plate limit (i.e., as $t \to 0$). To see this, we consider a heuristic example.

Example 1

Assume a four-node isoparametric quadrilateral element and, for simplicity, assume the element is of rectangular plan and the sides are aligned with the global x- and y-axes. In this case, the element expansions may be written as

$$w^h = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 xy \tag{5.3.1}$$

$$\theta^h_\alpha = \gamma_{\alpha 0} + \gamma_{\alpha 1} x + \gamma_{\alpha 2} y + \gamma_{\alpha 3} xy \tag{5.3.2}$$

where β_i and $\gamma_{\alpha i}$, $0 \le i \le 3$, are constants that depend upon the nodal parameters w^h_a and $\theta^h_{\alpha a}$, $1 \le a \le 4$, respectively. The conditions

$$\begin{aligned} 0 &= \gamma_1 \\ &= -\theta^h_1 + w^h_{,1} \\ &= (-\gamma_{10} + \beta_1) - \gamma_{11} x + (-\gamma_{12} + \beta_3) y - \gamma_{13} xy \end{aligned} \tag{5.3.3}$$

$$\begin{aligned} 0 &= \gamma_2 \\ &= -\theta^h_2 + w^h_{,2} \\ &= (-\gamma_{20} + \beta_2) + (-\gamma_{21} + \beta_3) x - \gamma_{22} y - \gamma_{23} xy \end{aligned} \tag{5.3.4}$$

impose eight constraints per element and are approximately in force as $t \to 0$ if exact integration of $\boldsymbol{k}^e_s$ is performed. (Two-by-two Gauss integration is exact in this case.) In a large rectangular mesh, there are approximately three degrees-of-freedom per element, and thus the element tends to be overly constrained. In practice, worthless numerical results are obtained [1]. To alleviate the "locking" effect, one might consider using one-point Gauss quadrature for $\boldsymbol{k}^e_s$. Clearly, this results in only two constraints per element, and now there are more degrees of freedom than there are constraints. This element, with one-point shear integration and 2×2 bending integration, was proposed and shown to be effective by Hughes et al. [2].

Arguments similar to those in Example 1 have been used to evaluate other possibilities (see Pugh et al. [1, 3] and Malkus and Hughes [4]).

The situation is seen to be similar to that for the incompressible problem discussed in Chapter 4. Again we shall define the ***constraint ratio***, r, for the standard mesh of Fig. 4.3.3 by

$$r = \frac{n_{eq}}{n_c}$$

where, in the present case, n_{eq} is the total number of displacement and rotation equations after boundary conditions have been imposed and n_c is the total number of shear strain constraints. Again, the idea is that as the number of equations in the standard mesh approaches infinity, r should approximate the ratio of equilibrium equations to constraints for the governing system of partial differential equations (in the present case, 3 and 2, respectively). Consequently, here the ideal value of r would be $\frac{3}{2}$. Smaller values would indicate the presence of too many shear strain constraints and a potential for locking. A larger value would indicate too few shear strain constraints and suggest that the Kirchhoff limit might be poorly approximated. Note that for the fully integrated four-node element discussed above, $r = \frac{3}{8}$, indicative of locking, whereas if one-point Gaussian quadrature is used for $\boldsymbol{k}_s^e$, then $r = \frac{3}{2}$, the optimal value.

We wish to emphasize again that the constraint ratio is only a quick device for estimating an element's propensity to lock. (See the discussion in Sec. 4.3.7.) In fact, the constraint ratio is not as successful for plates as for incompressible continuum elements. (There are excellent plate elements with constraint ratios less than $\frac{3}{2}$. See, for example, Sec. 5.3.7.) A superior, yet still simple, methodology for assessing the tendency of plates to lock is based upon the ***Kirchhoff mode concept*** [17, 18]. Much of the recent work on plate element design explicitly or implicitly employs this concept. The interested reader should consult [17] for a complete description.

5.3.3 Boundary Conditions

It is important to realize that boundary conditions in the present theory are not always the same as those for the classical thin plate theory. The differences occur in the specification of the "simply supported" case. In the present theory, there are two ways of going about this, depending on the actual physical constraint. Rather than being an additional complication, this freedom turns out to be a considerable benefit, for it enables the solution of problems in which thin plate finite elements have heretofore failed (see Rossow [5] and Scott [6]).

Consider a smooth portion of the plate boundary and a local s, n-coordinate system to it (s denotes the tangential direction and n the outward normal direction; see Fig. 5.3.1). The most common boundary conditions encountered in practice are given as follows.

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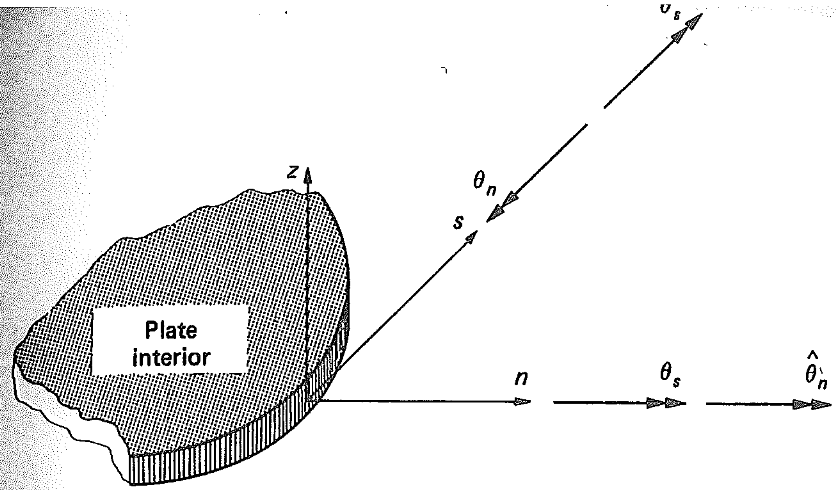


Figure 5.3.1 Local tangential-n
coordinate system at plate bound

Clamped

$$w = 0$$
$$\theta_s = 0$$
$$\theta_n = 0$$

Free

$$Q = 0$$
$$M_s = 0$$
$$M_n = 0$$

Simply supported

SS_1 $\quad w = 0$

$$M_s = 0$$
$$M_n = 0$$

SS_2 $\quad w = 0$

$$\theta_s = 0$$
$$M_n = 0$$

Skew Symmetric
$W = 0$
$\theta_s = 0$
$M_n = 0$

Symmetric

$$Q = 0$$
$$M_s = 0$$
$$\theta_n = 0$$

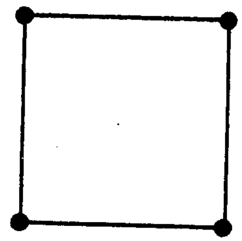
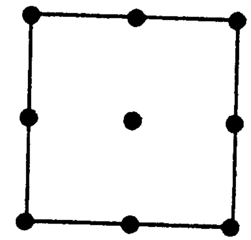
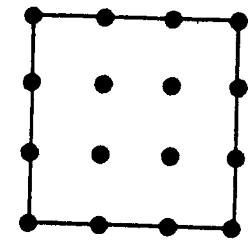
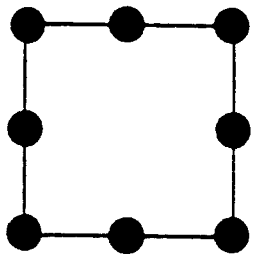
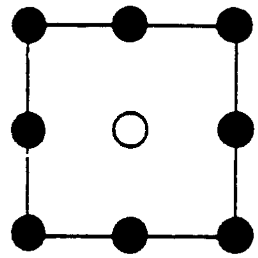
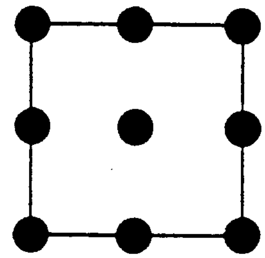
			
w, θ_1, θ_2 shape functions	Bilinear	Biquadratic	Bicubic
Uniform reduced integration	1×1 *U*1	2×2 *U*2	3×3 *U*3
Selective reduced integration	1×1 shear 2×2 bending *S*1	2×2 shear 3×3 bending *S*2	3×3 shear 4×4 bending *S*3

Figure 5.3.3 Lagrange plate elements. Three degrees of freedom per node: w, θ_1, θ_2.

	Serendipity 	Heterosis 	Lagrange 
w-shape functions	Serendipity	Serendipity	Lagrange
θ_1, θ_2-shape functions	Serendipity	Lagrange	Lagrange
Integration scheme	$U2$	$S2$	$S2$
Number of spurious zero-energy modes	1*	0	1
Constraint ratio	1.125	1.375	1.5

Key:

● w, θ_1, θ_2 degrees of freedom

○ θ_1, θ_2 degrees of freedom

$U2 = 2 \times 2$ Gauss

$$S2 = \begin{cases} \text{bending } 3 \times 3 \text{ Gauss} \\ \text{shear } \quad 2 \times 2 \text{ Gauss} \end{cases}$$

*Not communicable in a mesh of two or more elements.